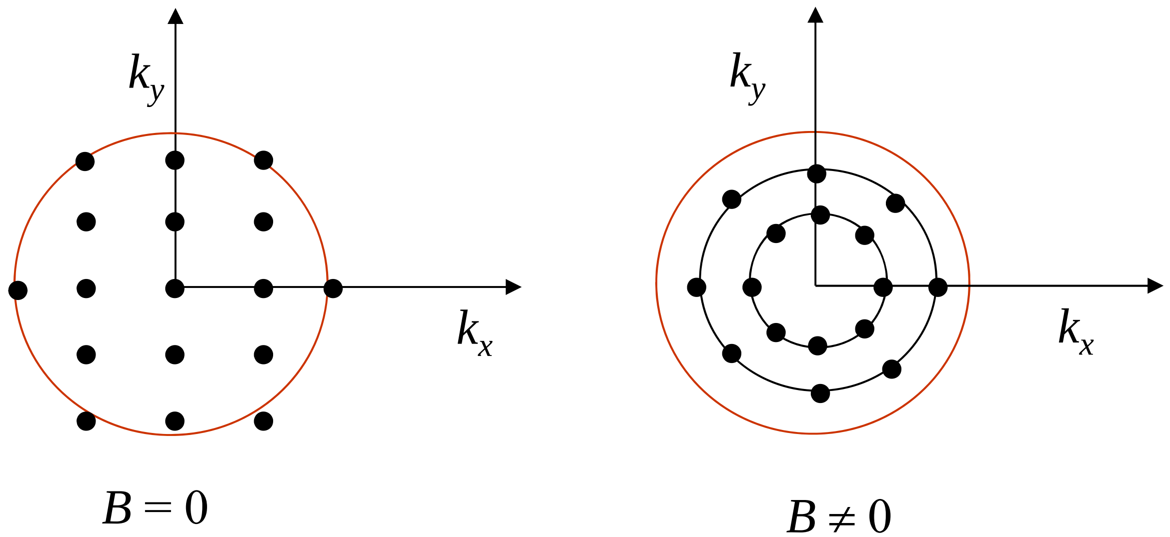
12. Free Electrons in a Magnetic Field

Nov 11, 2019

Landau levels



The number of solutions is conserved

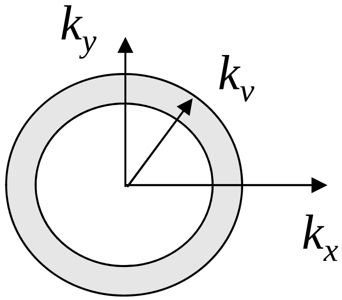
$$\psi = e^{ik_y y}\phi(x - x_0). \qquad x_0 = \frac{\hbar k_y}{qB_z}$$

In 2-D, the *k*-volume per *k* state is: $\left(\frac{2\pi}{L}\right)^2$

Density of states 2D

$$E_v = \hbar\omega_c\left(v + \tfrac{1}{2}\right)$$

The number of states between ring *v*-1 and ring *v* is



$$\frac{\pi\left(k_v^2 - k_{v-1}^2\right)}{\left(\dfrac{2\pi}{L}\right)^2}$$

$$\frac{\hbar^2 k_v^2}{2m} = \hbar\omega_c\left(v + \tfrac{1}{2}\right)$$

$$k_v^2 - k_{v-1}^2 = \frac{2m\omega_c}{\hbar}\left[\left(v + \tfrac{1}{2}\right) - \left(v - 1 + \tfrac{1}{2}\right)\right] = \frac{2m\omega_c}{\hbar}$$

The number of states between ring *v*-1 and ring *v* is $\dfrac{m\omega_c}{2\pi\hbar}L^2$

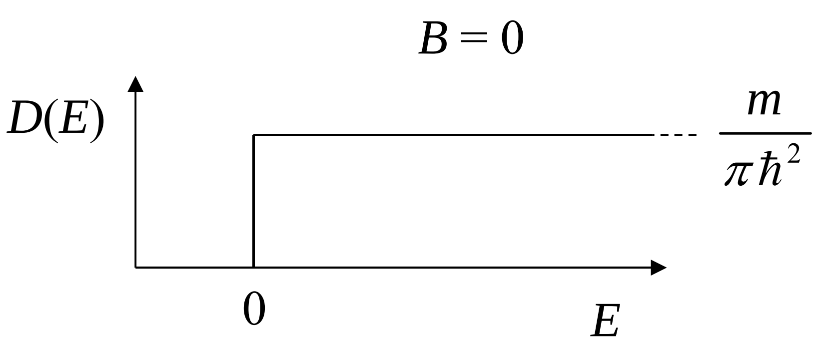
The density of states per spin is $\dfrac{m\omega_c}{2\pi\hbar}$

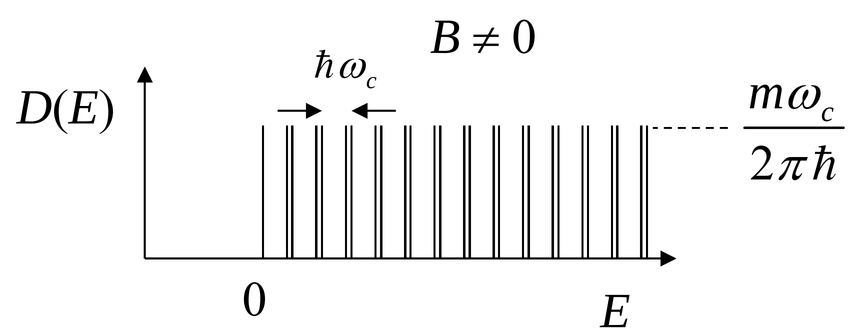
Spin

In a magnetic field, there is a shift of the energy of the electrons because of their spin.

$$E = -\vec{\mu}\cdot\vec{B} = \pm\frac{g}{2}\mu_B B$$

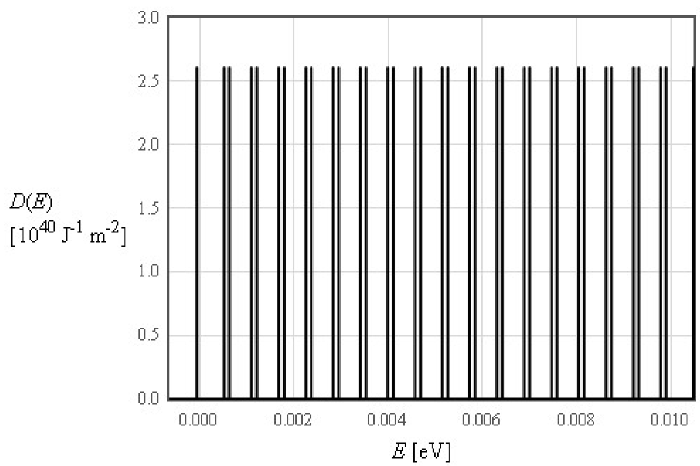
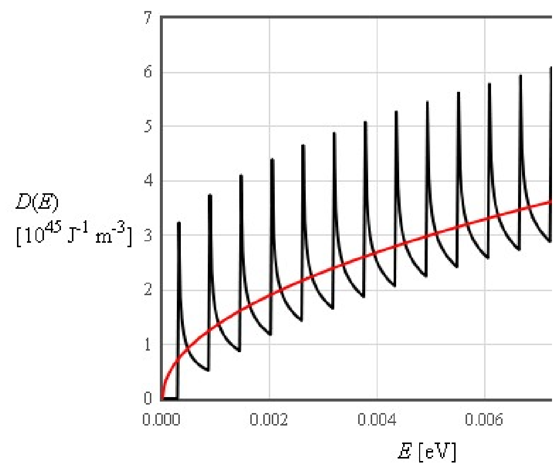
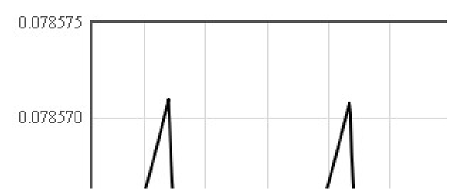
Bohr magneton $\mu_B = \dfrac{e\hbar}{2m_e}$ g-factor $g \approx 2$ $\hbar\omega_c = \dfrac{\hbar e B}{m} = 2\mu_B B$





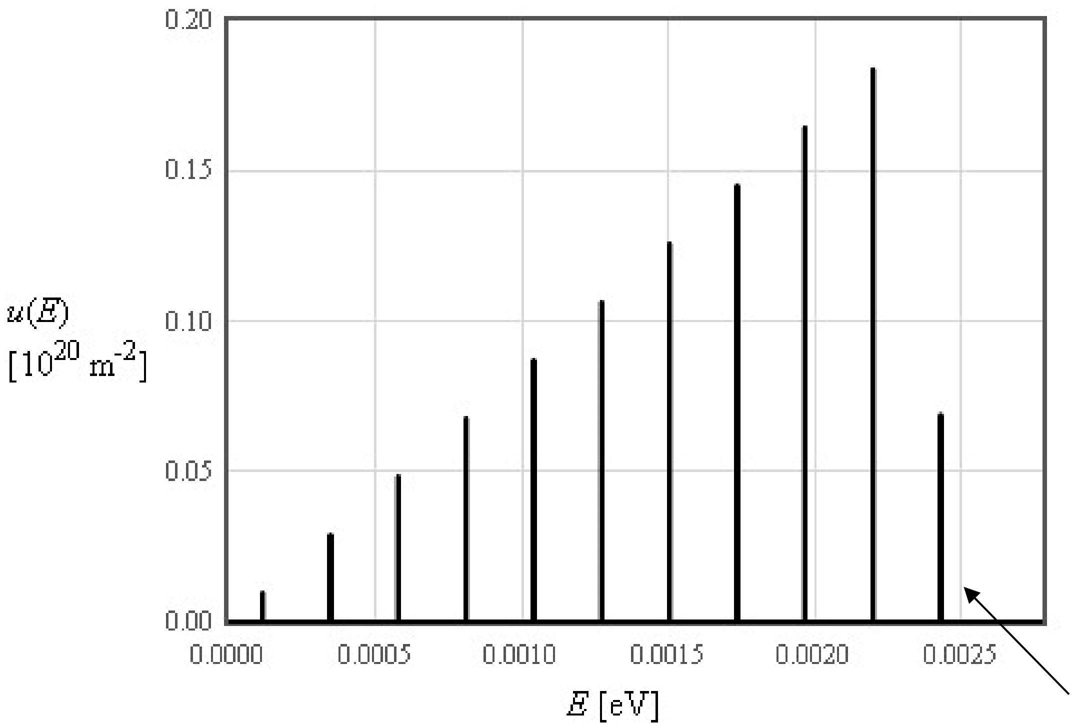
$$D(E) = \frac{m\omega_c}{2\pi\hbar}\sum_{\nu=0}^{\infty}\delta\left(E - \hbar\omega_c\left(\nu + \tfrac{1}{2} - \tfrac{g}{4}\right)\right) + \delta\left(E - \hbar\omega_c\left(\nu + \tfrac{1}{2} + \tfrac{g}{4}\right)\right)$$

ization of the Schrödinger equation for free electrons a magnetic field in 2 and 3 dimensions.

	2-D Schrödinger equation $i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$	3-D Schrödinger equation $i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$
Eigenfunction solutions	$\psi=g_v(x)\exp\left(ik_y y\right)$ $g_v(x)$ is a harmonic oscillator wavefunction	$\psi=g_v(x)\exp\left(ik_y y\right)\exp\left(ik_z z\right)$ $g_v(x)$ is a harmonic oscillator wavefun
Energy eigenvalues	$E=\hbar\omega_c\left(v+\tfrac{1}{2}\right)$ J $v=0,1,2,\cdots \qquad \omega_c=\frac{\lvert eB_z\rvert}{m}$	$E=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_c\left(v+\tfrac{1}{2}\right)$ J $v=0,1,2,\cdots \qquad \omega_c=\frac{\lvert eB_z\rvert}{m}$
Density of states	$D(E)=\frac{m\omega_c}{2\pi\hbar}\sum_{v=0}^{\infty}\delta\left(E-\hbar\omega_c\left(v+\tfrac{1}{2}\right)-\frac{g\mu_B}{2}B\right)+\delta\left(E-\hbar\omega_c\left(v+\tfrac{1}{2}\right)+\frac{g\mu_B}{2}B\right)$ $\mathrm{J^{-1}m^{-2}}$  Calculate DoS	$D(E)=\frac{(2m)^{3/2}\omega_c}{4\pi^2\hbar^2}\sum_{v=0}^{\infty}\frac{H\left(E-\hbar\omega_c\left(v+\frac{1}{2}\right)\right)}{\sqrt{E-\hbar\omega_c\left(v+\frac{1}{2}\right)}}$  Calculate DoS
	$E_F=\hbar\omega_c\left(\mathrm{Int}\left(\frac{\pi\hbar n}{m\omega_c}\right)+\frac{1}{2}\right)$ 	

Energy spectral density 2d

At $T = 0$

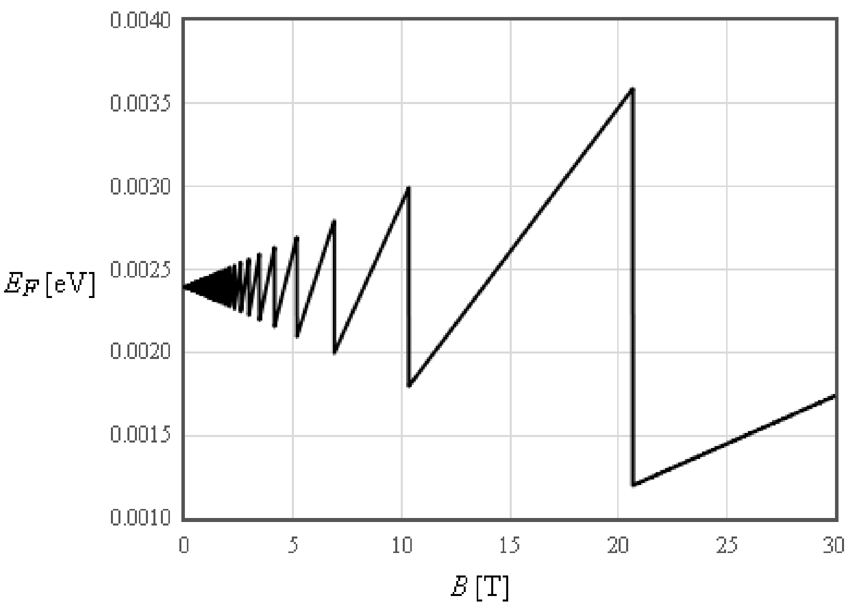


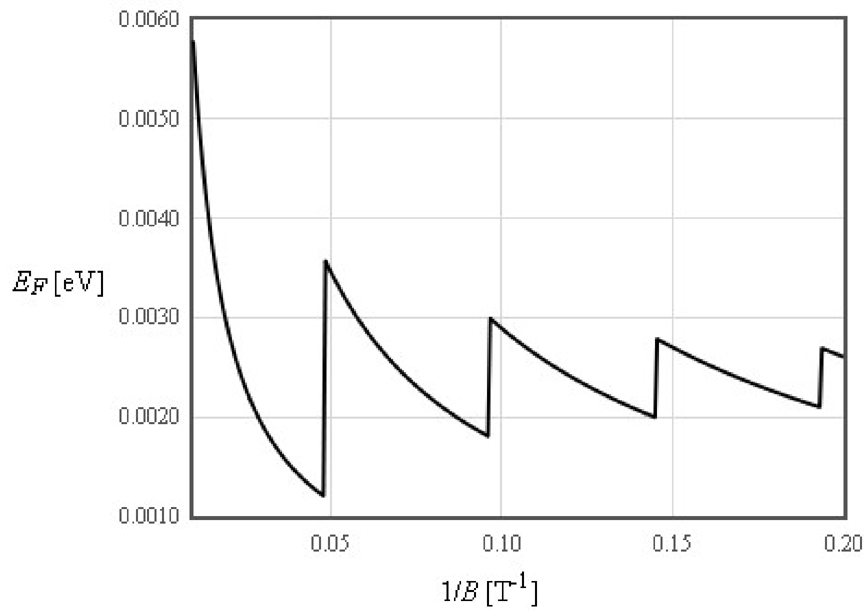
unfilled Landau level

analog to the Planck radiation law

Fermi energy 2d

$$n = \int_{-\infty}^{E_F} D(E)dE$$





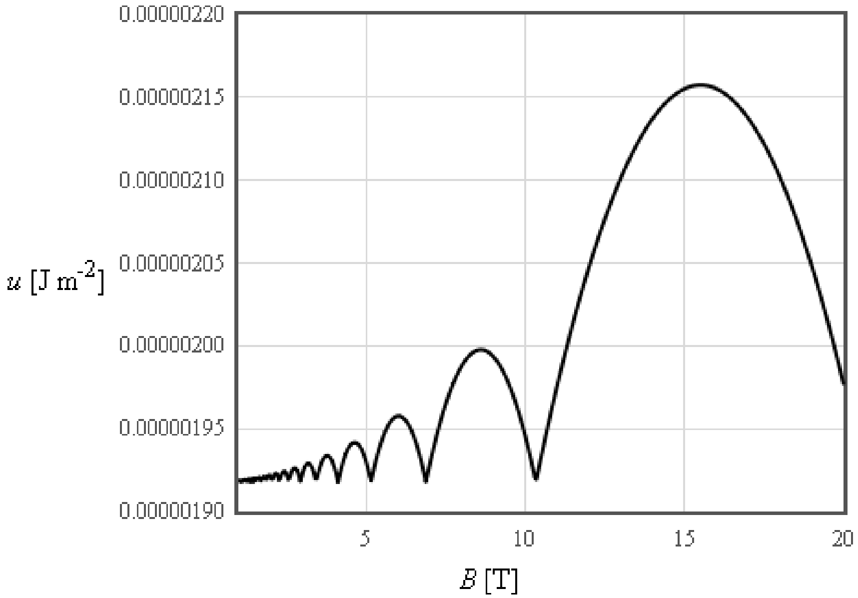
When there is only one Landau level, the Fermi energy rises linearly with field.

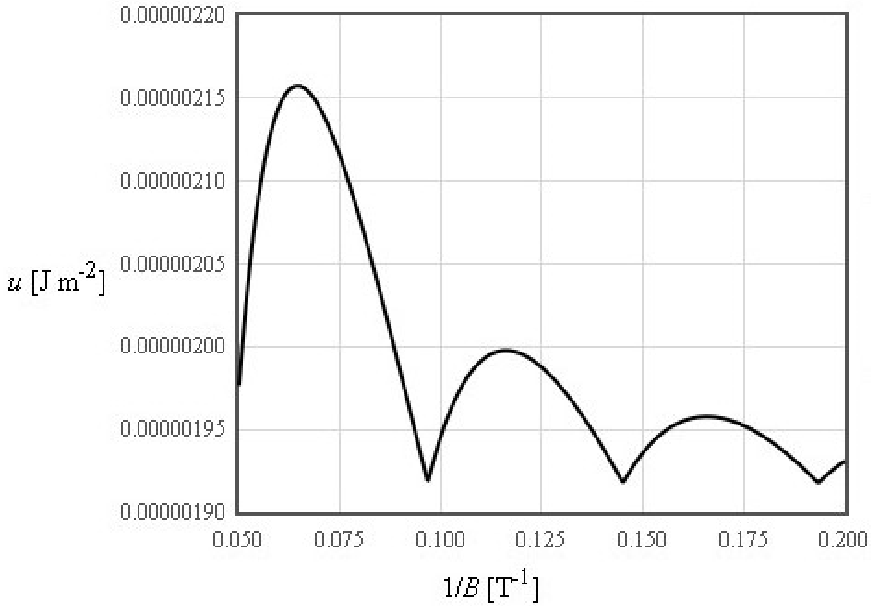
Periodic in $1/B$

Large field limit $\longrightarrow$ $E_F = \frac{\hbar\omega_c}{2} = \frac{\hbar eB}{2m}$

Internal energy 2d

At $T = 0$ $\quad u = \int_{-\infty}^{E_F} E D(E) dE$



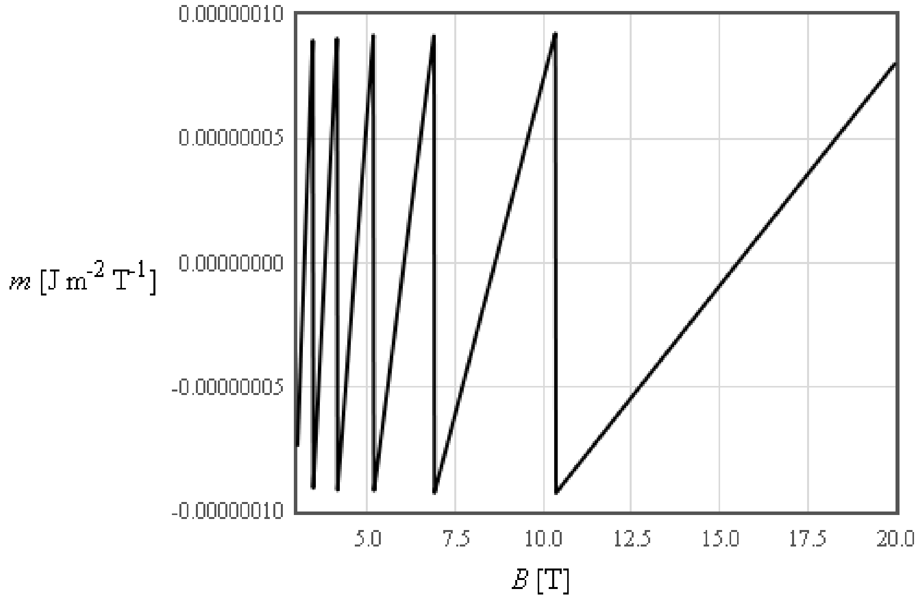


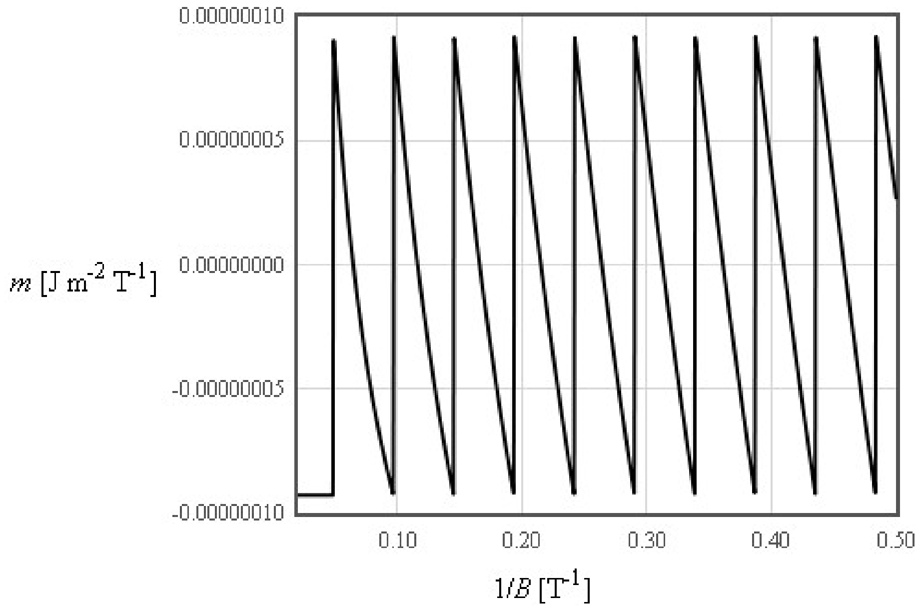
$$u = n\frac{\hbar\omega_c}{2} = n\frac{\hbar eB}{2m}$$

Large field limit

Magnetization 2d

At $T = 0$



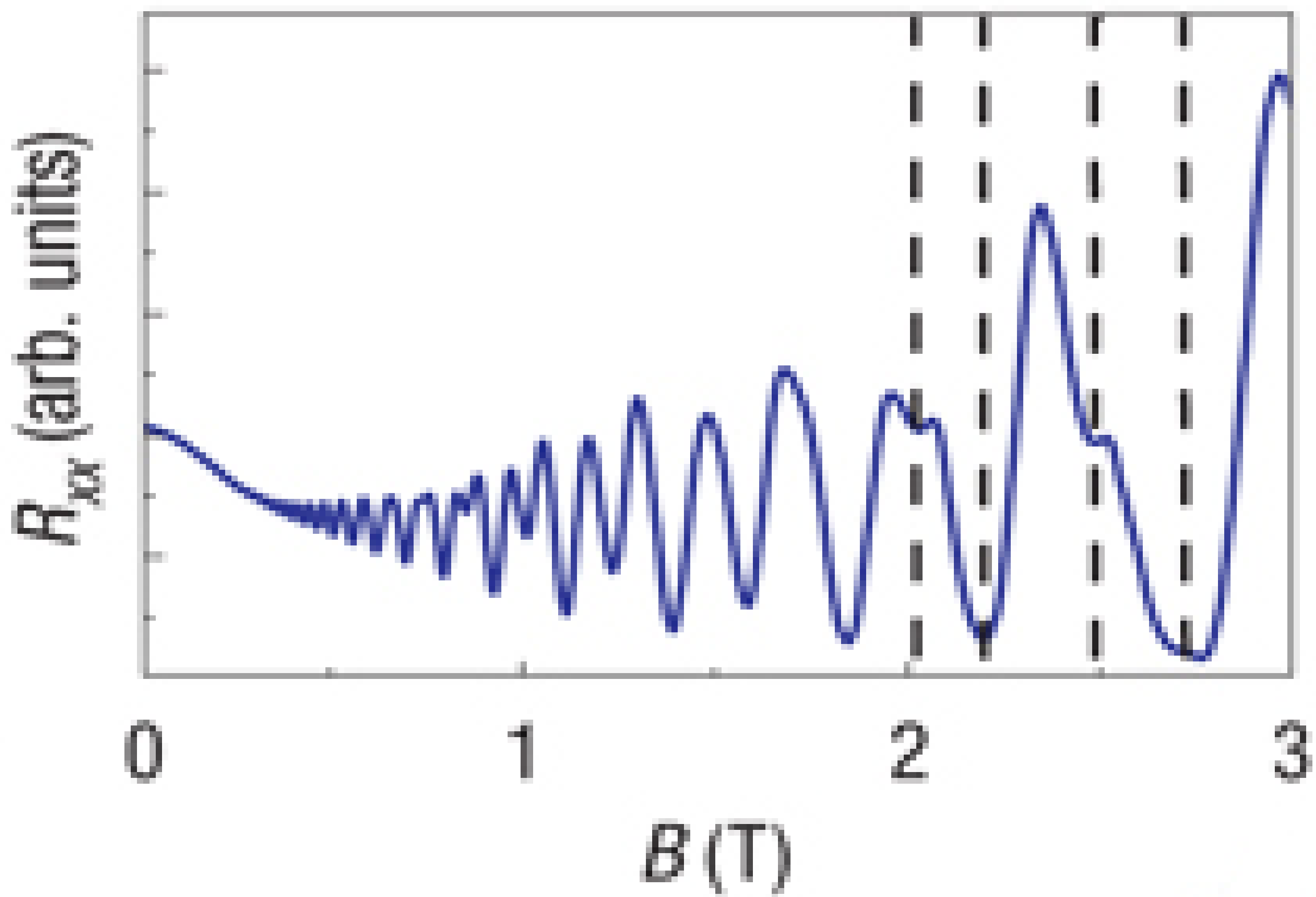


$$m = -\frac{du}{dB} = -n\frac{\hbar e}{2m}$$

Large field limit

de Haas - van Alphen oscillations

Shubnikov-De Haas oscillations



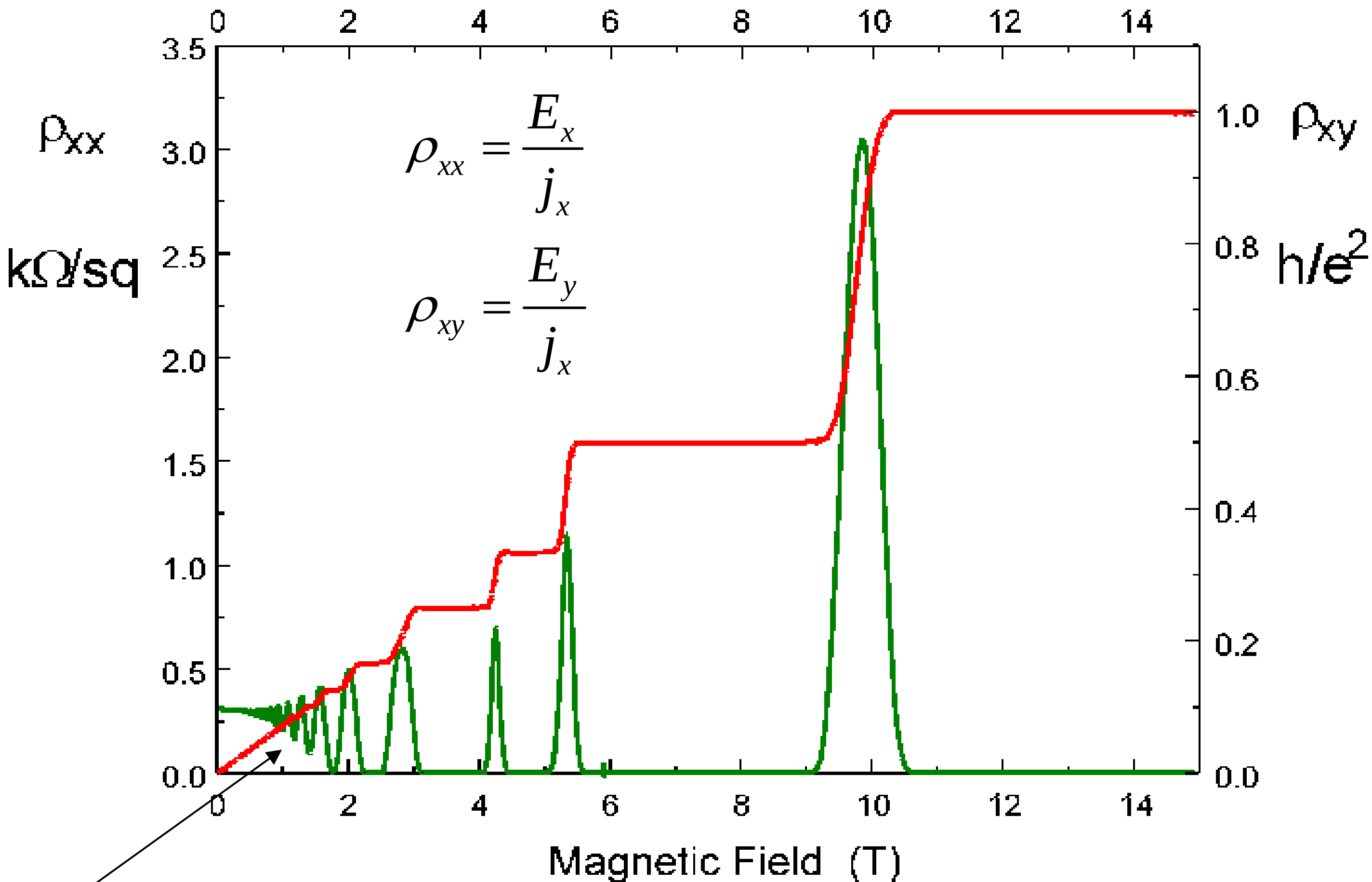
Scattering at the Fermi surface

At room temperature, phonon energies are much less than the Fermi energy. The energy of electrons hardly changes as they scatter from phonons. Electrons scatter from a point close to the Fermi surface to another point close to the Fermi surface.

Changing the magnetic field changes the number of states at the Fermi energy.

There are oscillations in the electrical conductivity as a function of magnetic field.

Quantum Hall Effect



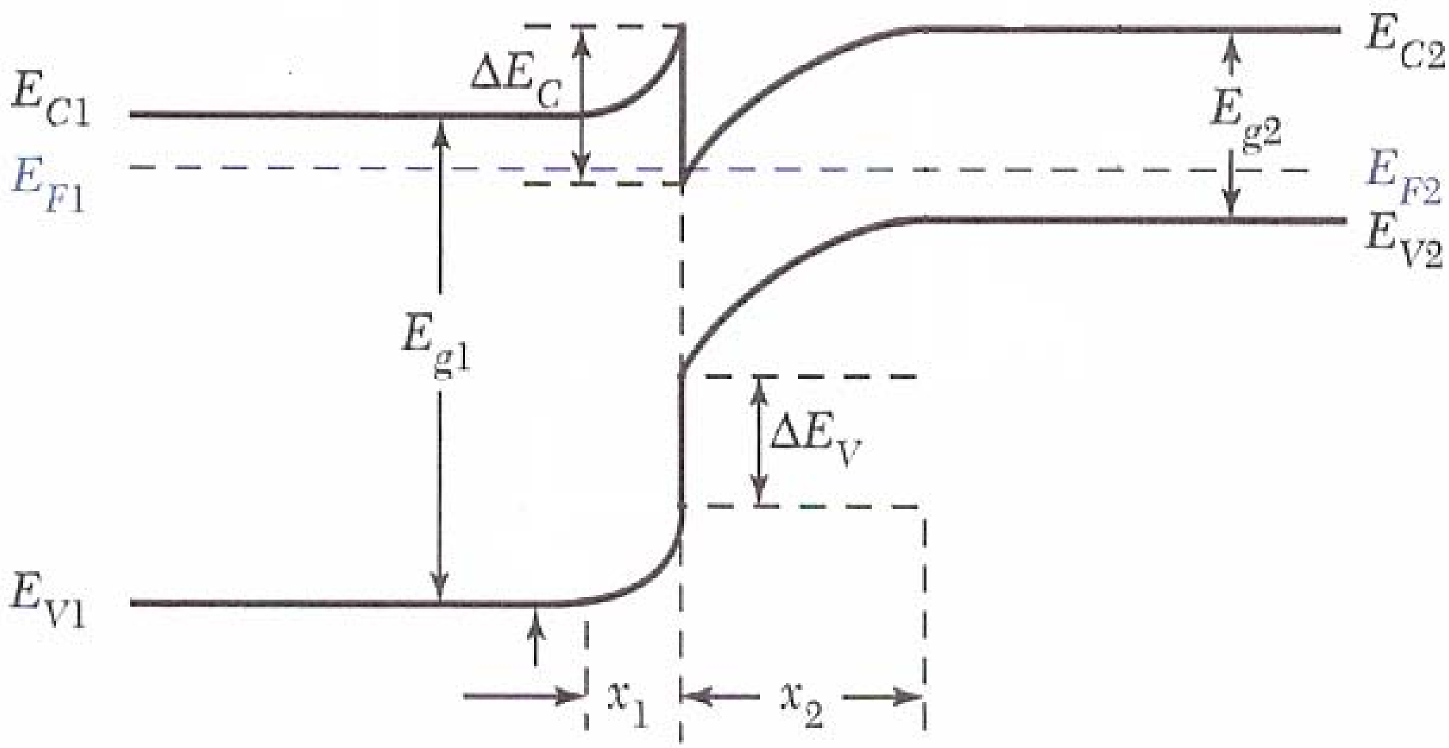
Klaus von Klitzing

Shubnikov-De Haas oscillations

Resistance standard
25812.807557(18) Ω

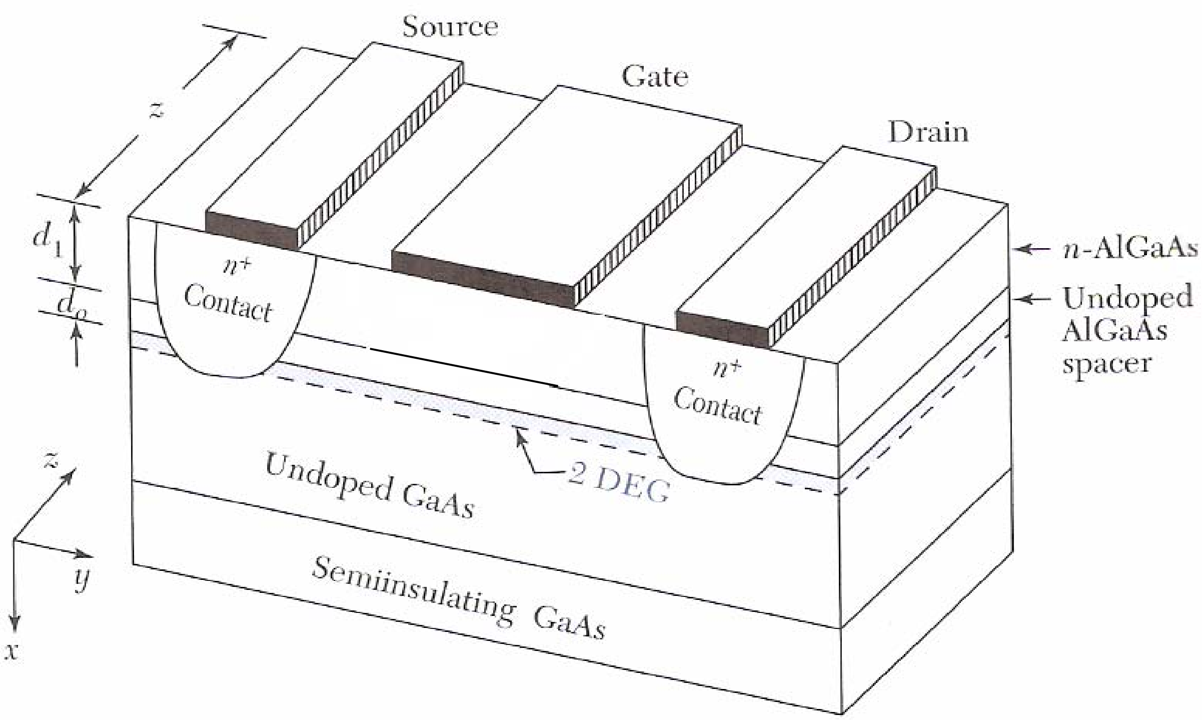
Heterostructure

pn junction formed from two semiconductors with different band gaps



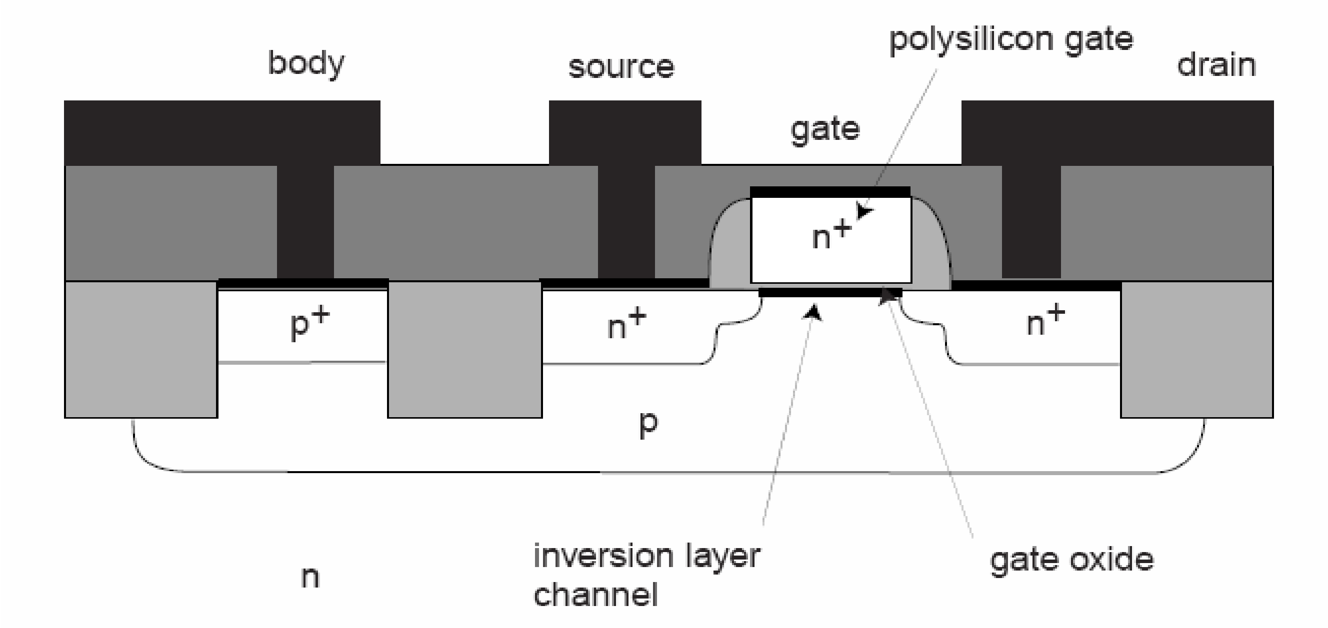
MODFET (HEMT)

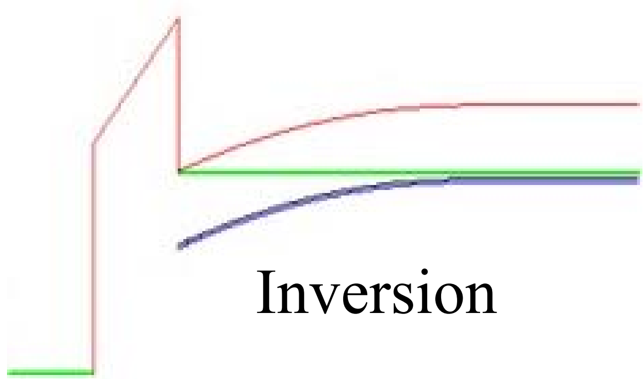
Modulation doped field effect transistor (MODFET)
High electron mobility transistor (HEMT)



The magnetic field can be at an angle to the 2DEG. The Landau splitting experiences the component perpendicular to the plane. The Zeeman splitting experiences the full field.

MOSFETs





Inversion

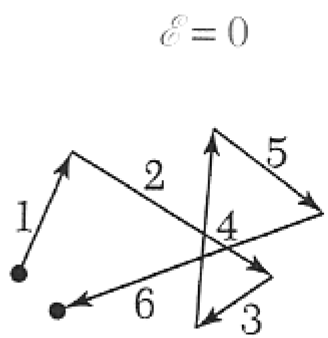
Drift

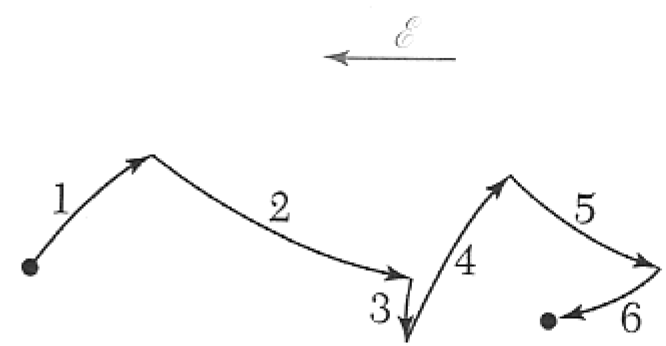
The electrons scatter and change direction after a time τ_{sc}.

Classical equipartition: $\frac{1}{2} m v_{th}^2 = \frac{3}{2} k_B T$

At 300 K, $v_{th} \sim 10^7$ cm/s.

mean free path: $\ell = v_{th} \tau_{sc} \sim 10$ nm ~ 200 atoms

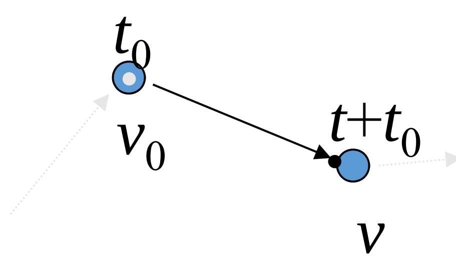




Drift (diffusive transport)

$$\vec{F} = -e\vec{E} = m^*\vec{a} = m^*\frac{d\vec{v}}{dt}$$

time between two collisions



$$\vec{v} = \vec{v}_0 - \frac{e\vec{E}}{m^*}(t - t_0)$$

$<v_0> = 0$ $\quad$ $<t - t_0> = \tau_{sc}$

$$\vec{v}_d = \frac{-e\vec{E}\tau_{sc}}{m^*} = \frac{-e\vec{E}\ell}{m^* v}$$

drift velocity: $\overrightarrow{v_{d,n}} = -\mu_n \overrightarrow{E}$ $\quad$ $\overrightarrow{v_{d,p}} = \mu_p \overrightarrow{E}$

Review of the Hall effect

$$\vec{F} = m\vec{a} = -e\vec{E} = m\frac{\vec{v}_d}{\tau_{\text{sc}}} \longleftarrow \quad \text{diffusive regime}$$

$$\vec{F} = -e\left(\vec{E} + \vec{v}\times\vec{B}\right) = m\frac{\vec{v}_d}{\tau_{\text{sc}}}$$

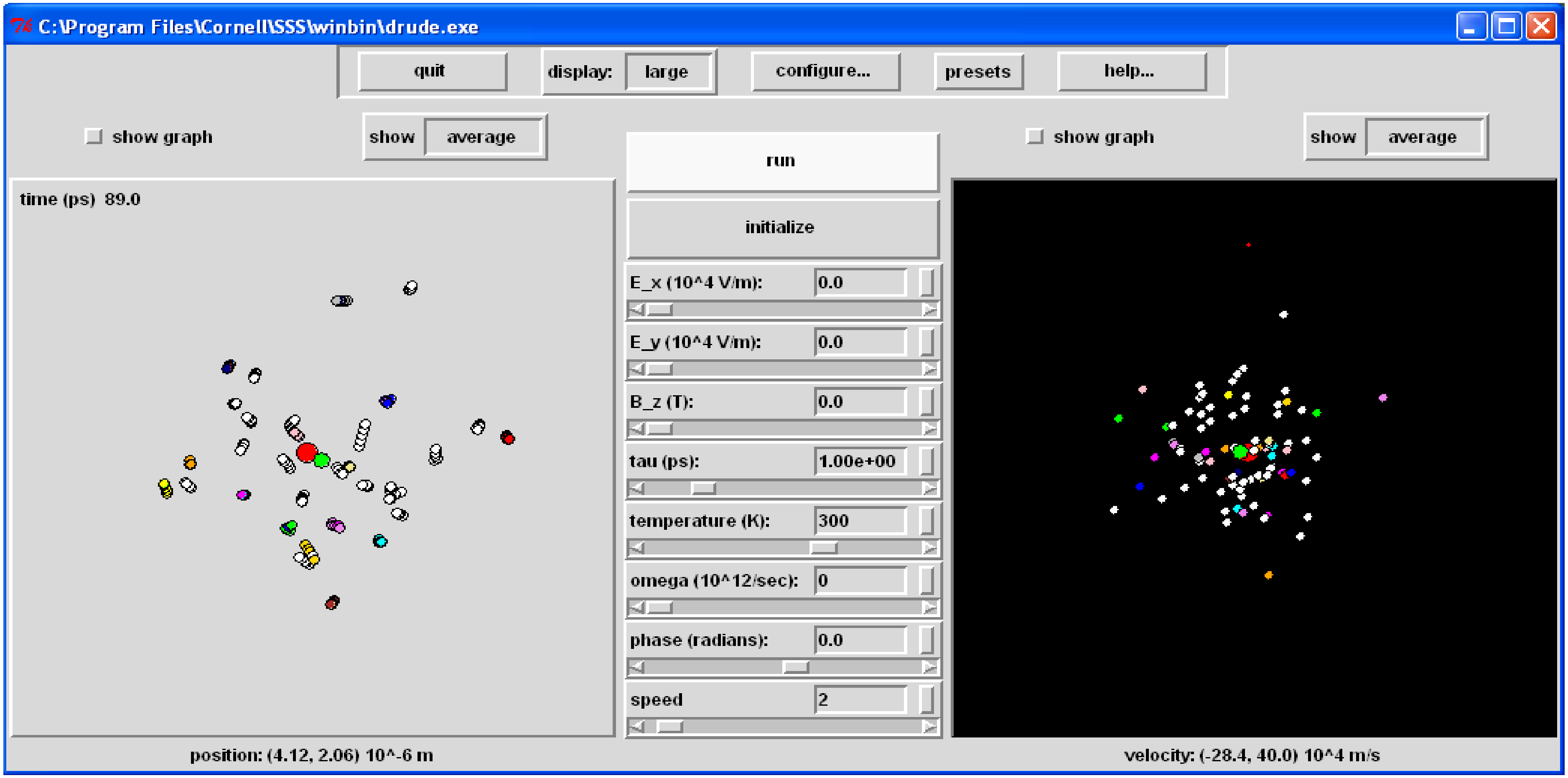
If B is in the z-direction, and E is in the x- direction, the three components of the force are

$$-e\left(E_x + v_{dy}B_z\right) = m\frac{v_{dx}}{\tau_{\text{sc}}}$$

$$ev_{dx}B_z = m\frac{v_{dy}}{\tau_{\text{sc}}} \quad \Rightarrow \quad \tan\theta_H = -\frac{eB_z}{m}\tau_{sc}$$

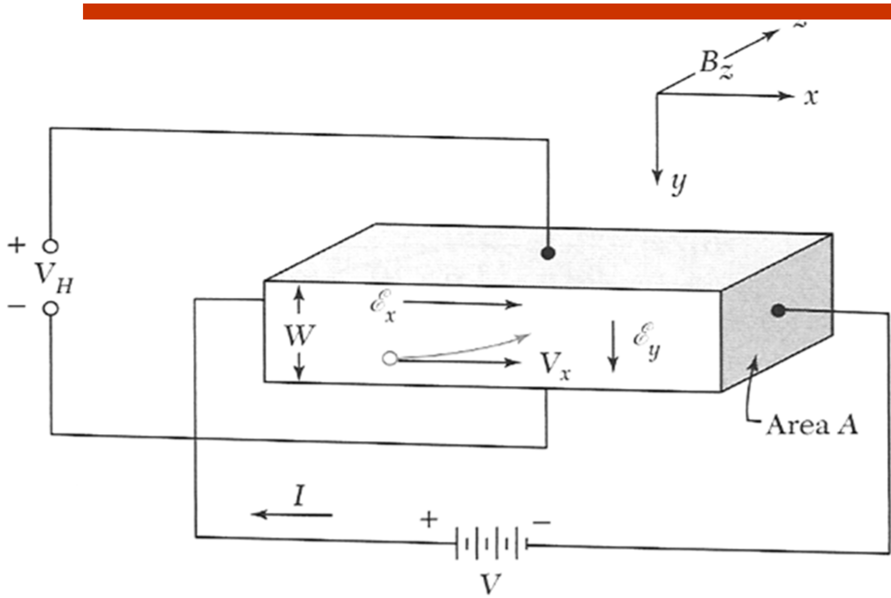
$$0 = m\frac{v_{dz}}{\tau_{\text{sc}}}$$

Hall angle



If no forces are applied, the electrons diffuse.
The average velocity moves against an electric field.
In just a magnetic field, the average velocity is zero.
In an electric and magnetic field, the electrons move in a straight line at the Hall angle.
The drift velocity decreases as the B field increases.

The Hall Effect (diffusive regime)



$$v_{d,x} = -\frac{eE_x\tau_{sc}}{m} - \frac{eB_z}{m}\tau_{sc}v_{d,y}$$

$$v_{d,y} = -\frac{eE_y\tau_{sc}}{m} + \frac{eB_z}{m}\tau_{sc}v_{d,x}$$

$$v_{d,z} = -\frac{eE_z\tau_{sc}}{m}$$

If $v_{d,y} = 0$,

$E_y = v_{d,x}B_z = V_H/W = R_H j_x B_z$ V_H = Hall voltage, R_H = Hall Constant

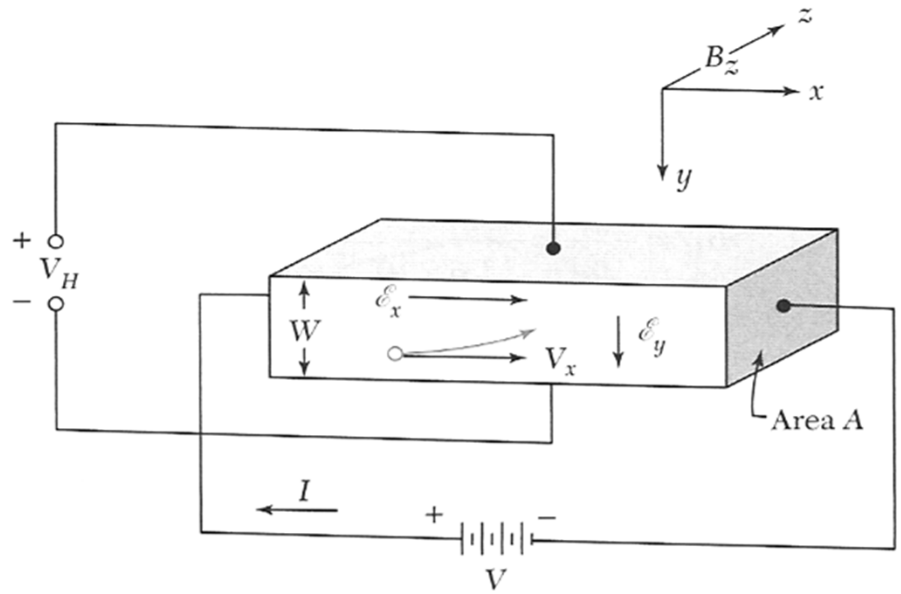
$v_{d,x} = - j_x/ne$

$R_H = E_y/j_x B_z = - 1/ne$

The Hall Effect (diffusive regime)

$$\rho_{xx} = \frac{E_x}{j_x}$$

$$\rho_{xy} = \frac{E_y}{j_x}$$



$R_H = E_y/j_xB_z = -1/ne$

multiply both sides by B_z

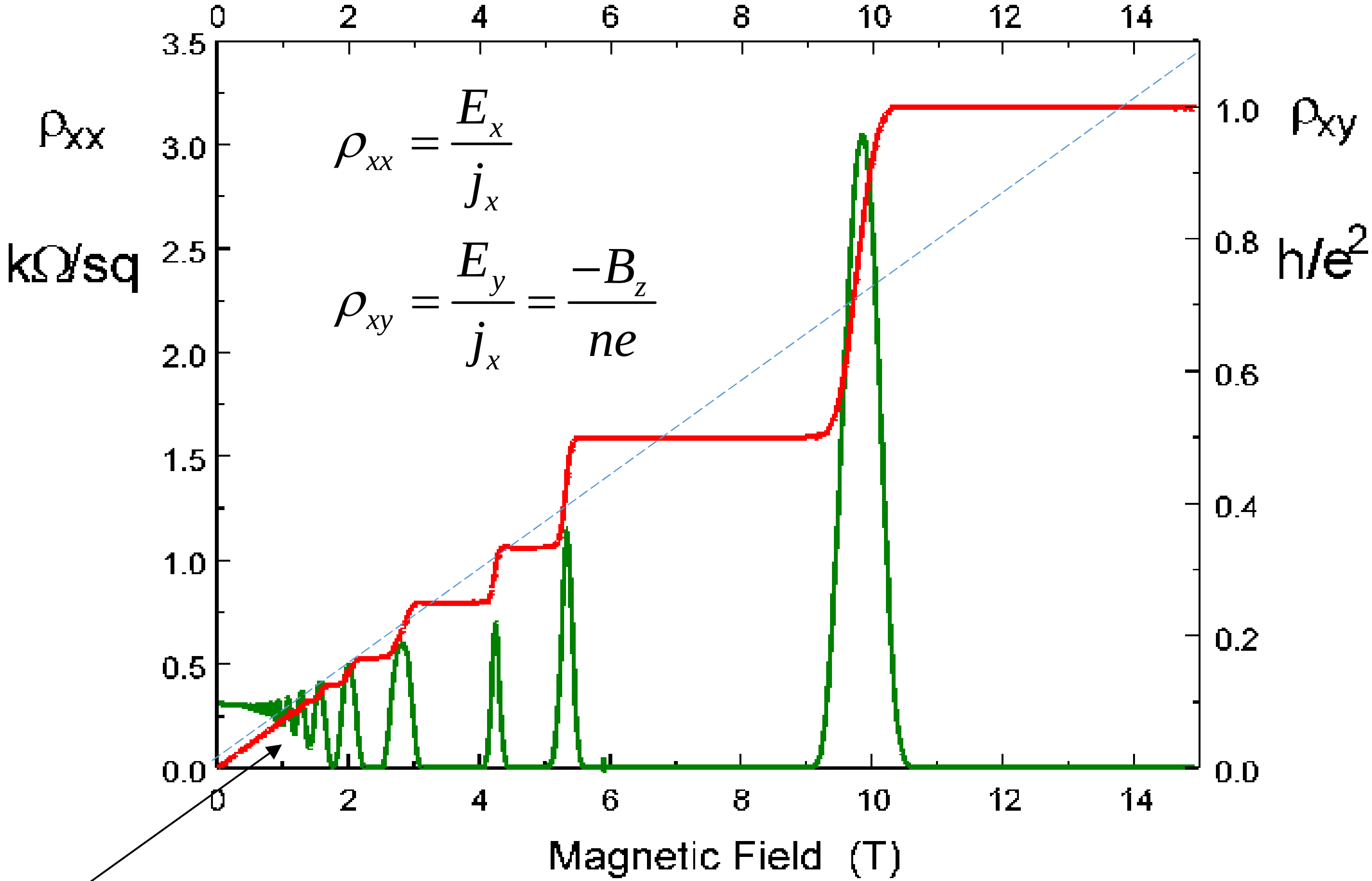
$$\rho_{xy} = \frac{E_y}{j_x} = \frac{-B_z}{ne}$$

In 2D, j has units of A/m and n has units of $1/m^2$.

In 3D, j has units of A/m^3 and n has units of $1/m^3$.

The Hall resistivity is proportional to the magnetic field.

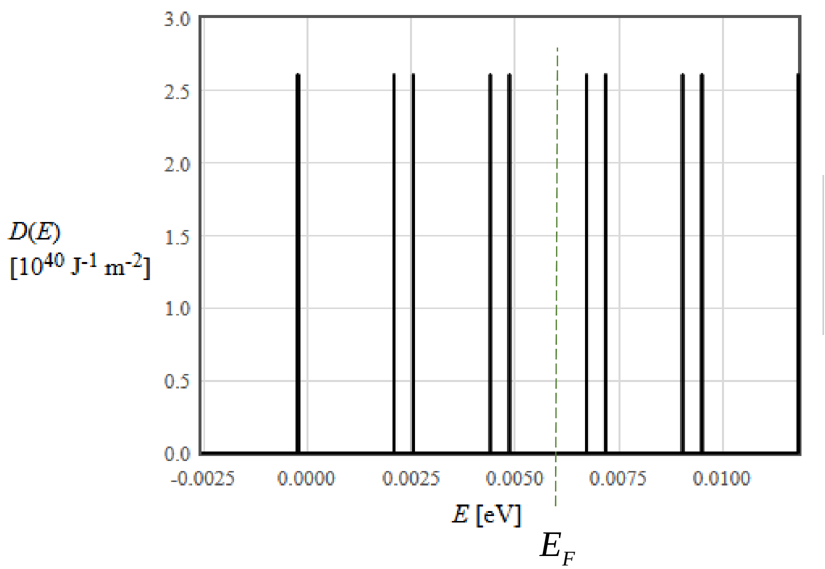
Quantum Hall Effect



Shubnikov-De Haas oscillations

Resistance standard
25812.807557(18) Ω

Quantum hall effect



If the Fermi energy is between Landau levels, the electron density n is an integer v times the degeneracy of the Landau level $n = D_0 v$

$$\rho_{xy} = \frac{E_y}{j_x} = \frac{-B_z}{ne}$$

Each Landau level can hold the same number of electrons.

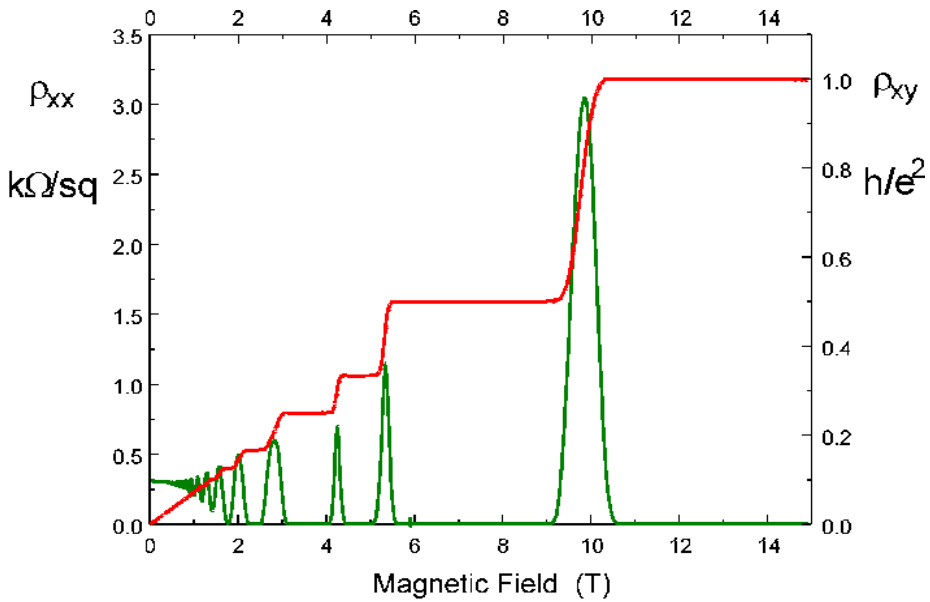
$$\rho_{xy} = \frac{-B_z}{ne} = \frac{-hD_0}{ve^2 D_0} = \frac{-h}{ve^2}$$

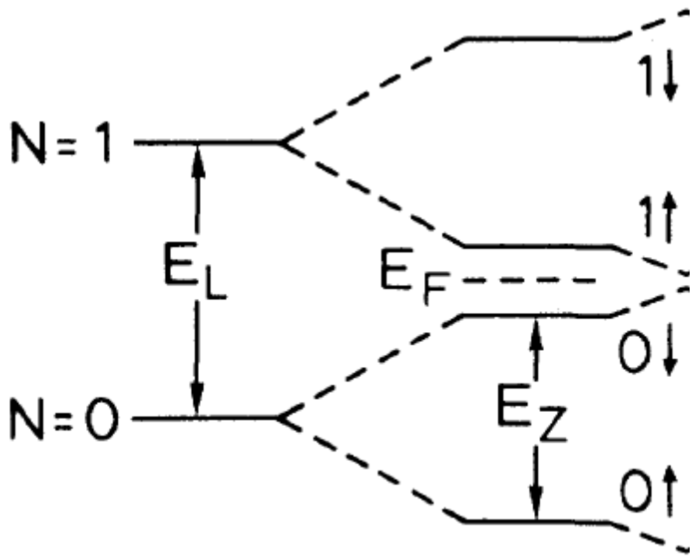
$$D_0 = \frac{m\omega_c}{2\pi\hbar} = \frac{eB_z}{h}$$

$$\omega_c = \frac{eB_z}{m} \qquad B_z = \frac{hD_0}{e}$$

Quantum hall effect

$$\rho_{xy} = \frac{h}{\nu e^2}$$



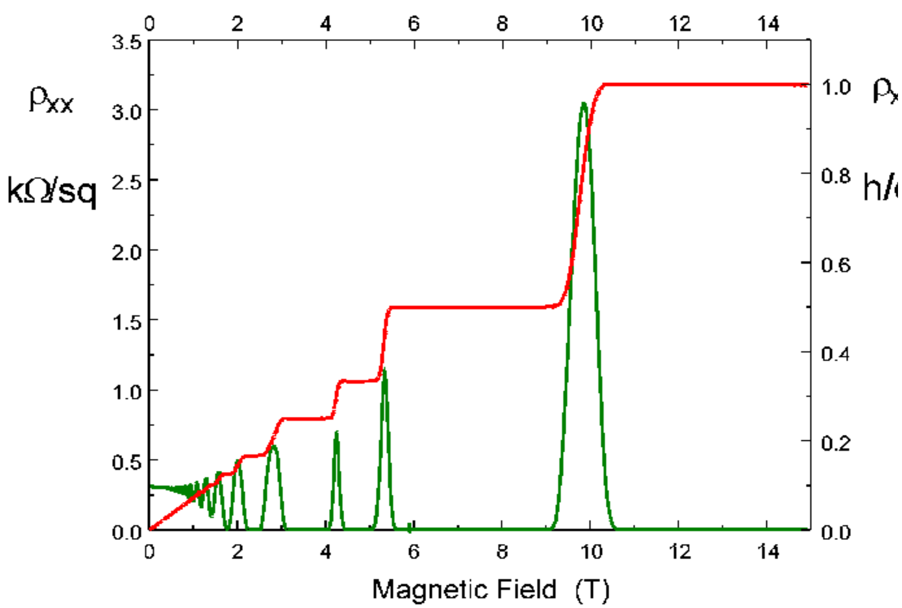


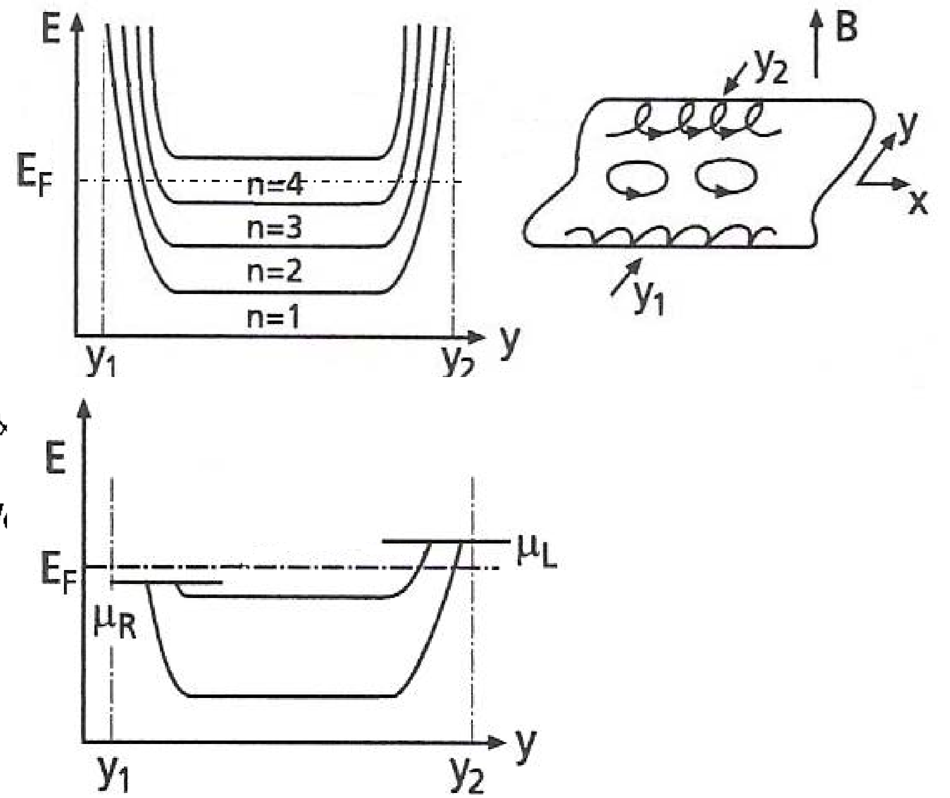
S. Koch, R. J. Haug, and K. v. Klitzing, Phys. Rev. B 47, 4048–4051 (1993)

Quantum Hall effect

Edge states are responsible for the zero resistance in ρ_{xx}

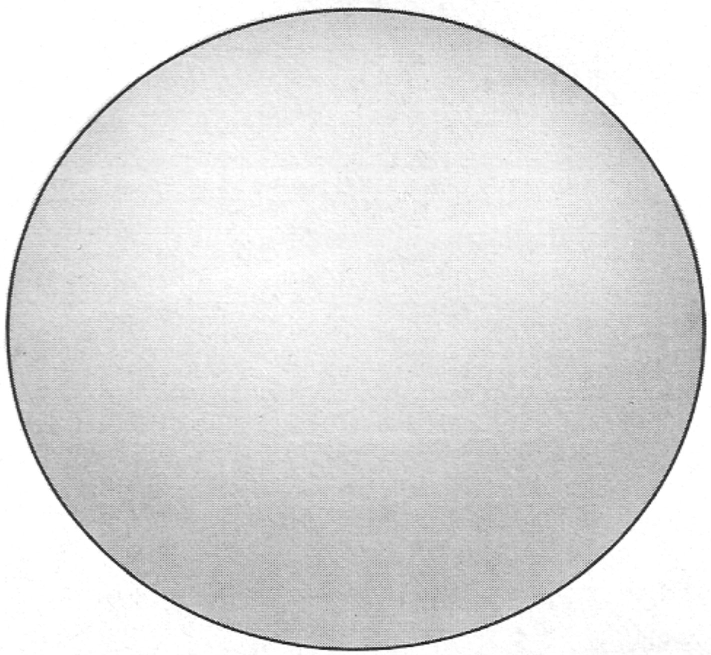
On the plateaus, resistance goes to zero because there are no states to scatter into.



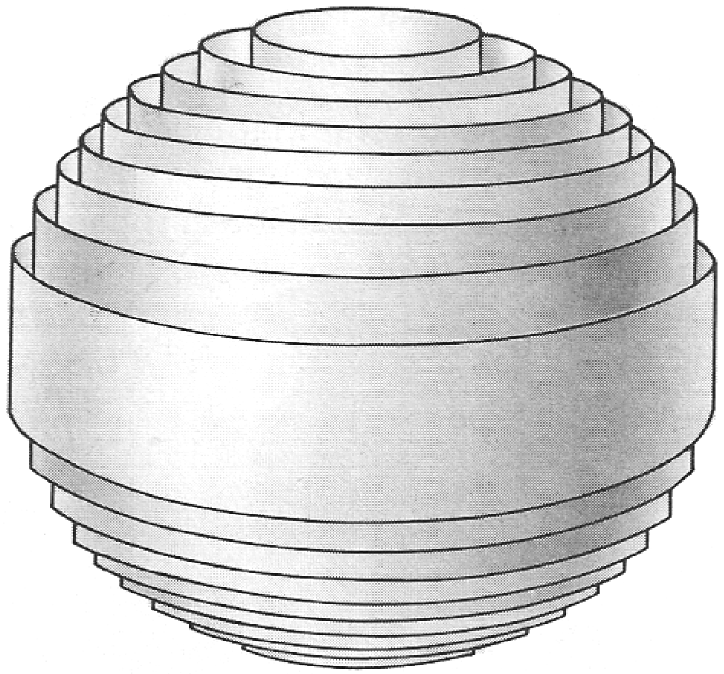


Ibach & Lueth (modified)

Fermi sphere in a magnetic field



$B = 0$

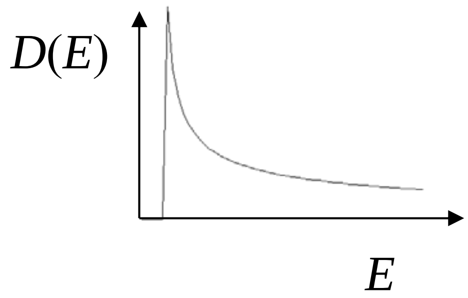


$B \neq 0$

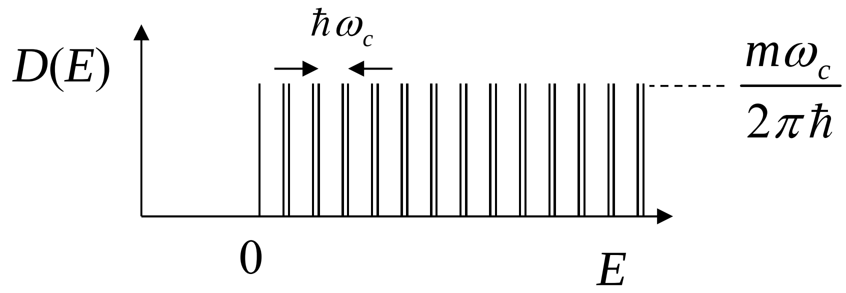
Landau cylinders

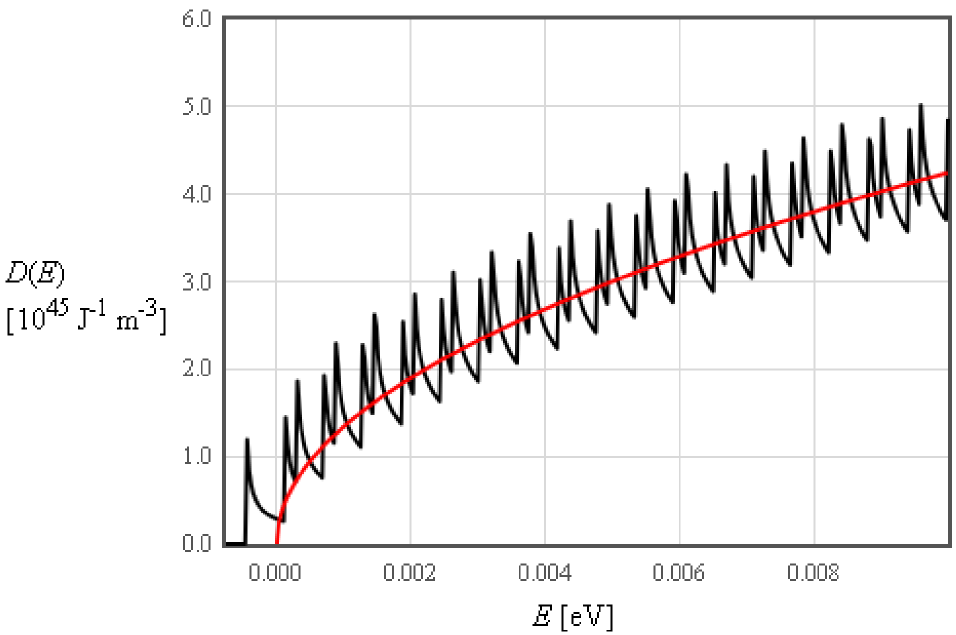
Density of states 3d

convolution of



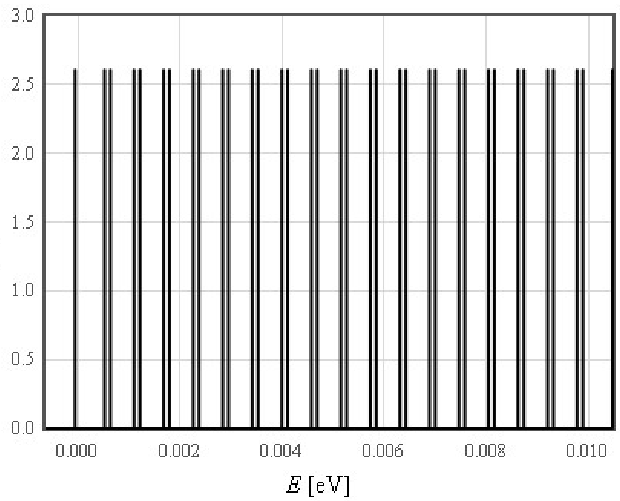
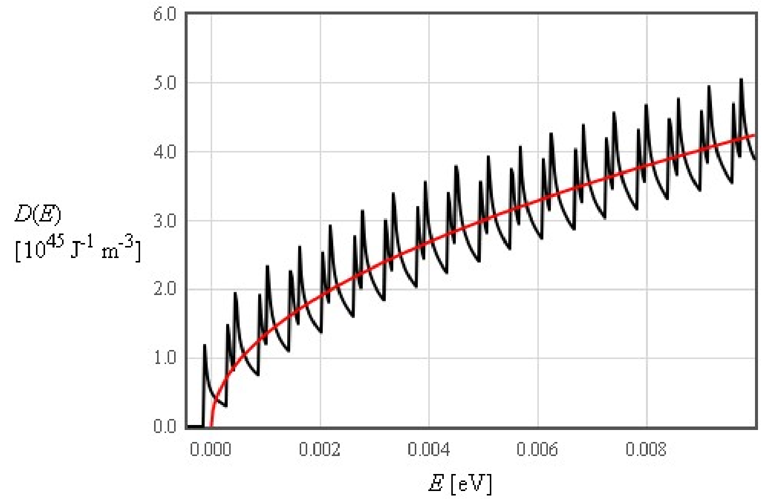
and





$$D(E) = \frac{(2m)^{3/2}\omega_c}{8\pi^2\hbar^2}\left(\sum_{\nu=0}^{\infty}\frac{H\left(E-\hbar\omega_c\left(\nu+\frac{1}{2}-g/4\right)\right)}{\sqrt{E-\hbar\omega_c\left(\nu+\frac{1}{2}-g/4\right)}}+\frac{H\left(E-\hbar\omega_c\left(\nu+\frac{1}{2}+g/4\right)\right)}{\sqrt{E-\hbar\omega_c\left(\nu+\frac{1}{2}+g/4\right)}}\right) \quad \text{J}^{-1}\text{m}^{-3}$$

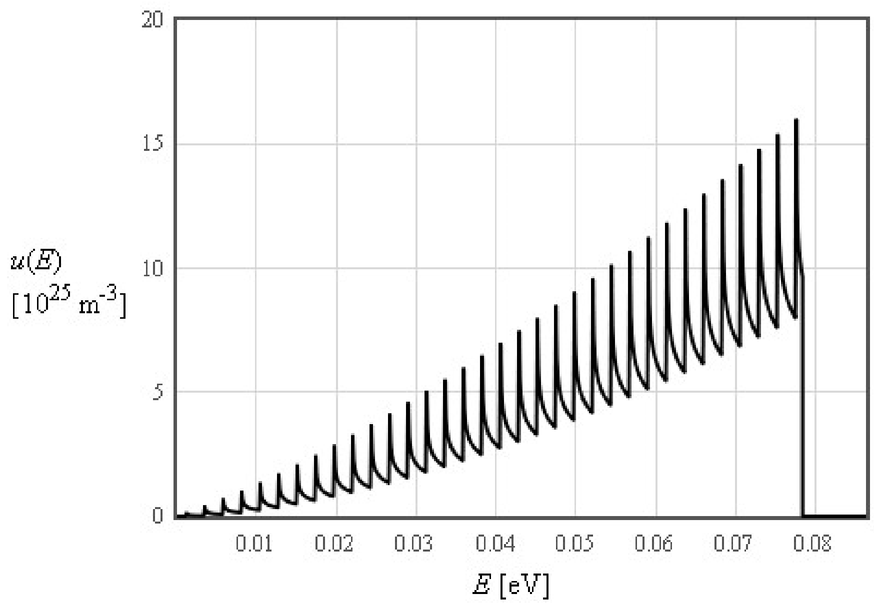
quation for free electrons a magnetic field in 2 and 3 dimensions.

2-D Schrödinger equation	3-D Schrödinger equation
$i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$	$i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$
$\psi=g_v(x)\exp\left(ik_y y\right)$ $g_v(x)$ is a harmonic oscillator wavefunction	$\psi=g_v(x)\exp\left(ik_y y\right)\exp\left(ik_z z\right)$ $g_v(x)$ is a harmonic oscillator wavefunction
$E=\hbar\omega_c\left(v+\frac{1}{2}\right)$ J $v=0,1,2,\cdots$ $\quad\omega_c=\frac{\lvert eB_z\rvert}{m}$	$E=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_c\left(v+\frac{1}{2}\right)$ J $v=0,1,2,\cdots$ $\quad\omega_c=\frac{\lvert eB_z\rvert}{m}$
$\sum_{v=0}^{\infty}\delta\left(E-\hbar\omega_c\left(v+\frac{1}{2}\right)-\frac{g\mu_B}{2}B\right)+\delta\left(E-\hbar\omega_c\left(v+\frac{1}{2}\right)+\frac{g\mu_B}{2}B\right)$ $\text{J}^{-1}\text{m}^{-2}$  Calculate DoS	$D(E)=\frac{(2m)^{3/2}\omega_c}{8\pi^2\hbar^2}\left(\sum_{v=0}^{\infty}\frac{H\left(E-\hbar\omega_c\left(v+\frac{1}{2}-g/4\right)\right)}{\sqrt{E-\hbar\omega_c\left(v+\frac{1}{2}-g/4\right)}}+\frac{H\left(E-\hbar\omega_c\left(v+\frac{1}{2}+g/4\right)\right)}{\sqrt{E-\hbar\omega_c\left(v+\frac{1}{2}+g/4\right)}}\right)$ $\text{J}^{-1}\text{m}^{-3}$  Calculate DoS
$E_F=\hbar\omega\left(\text{Int}\left(\frac{\pi\hbar n}{\cdots}\right)+\frac{1}{\cdots}\right)$	

Energy spectral density 3d

At $T = 0$

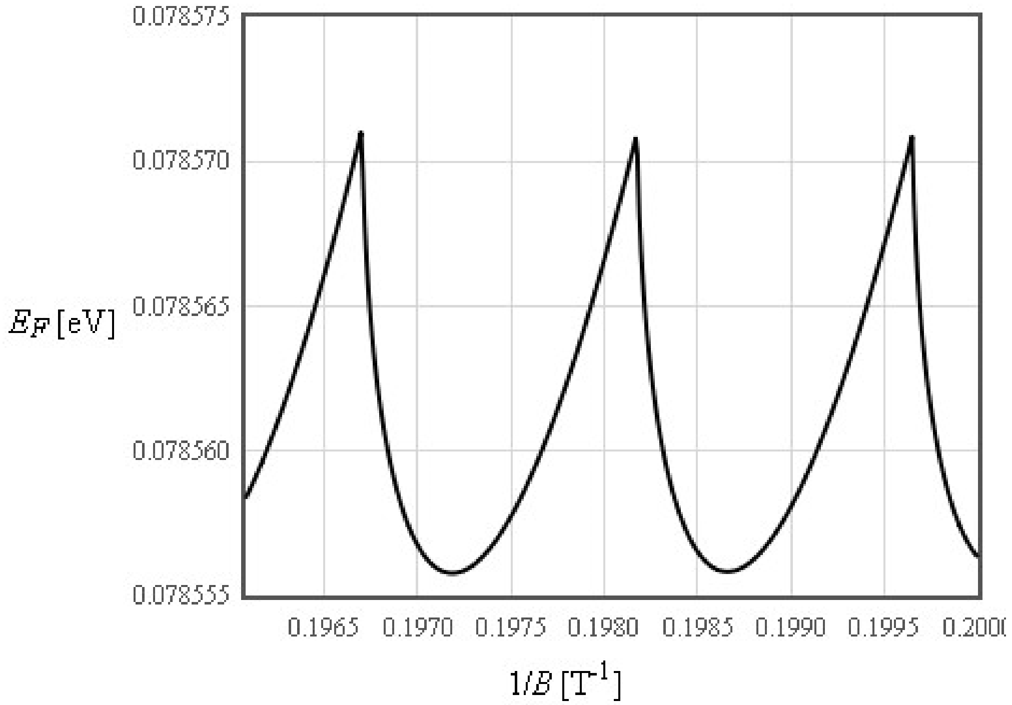
$$u(E) = ED(E)\,f(E)$$



$$u(T=0) = \int_{-\infty}^{E_F} ED(E)\,dE$$

Fermi energy 3d

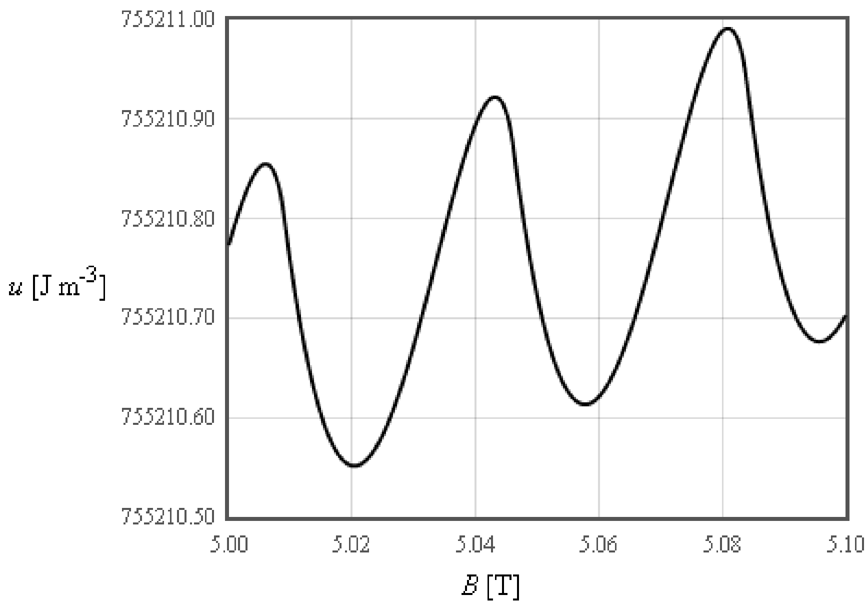
$$n = \int_{-\infty}^{E_F} D(E)\,dE$$



Periodic in 1/*B*

Internal energy 3d

$$u = \int_{-\infty}^{E_F} E D(E) dE$$



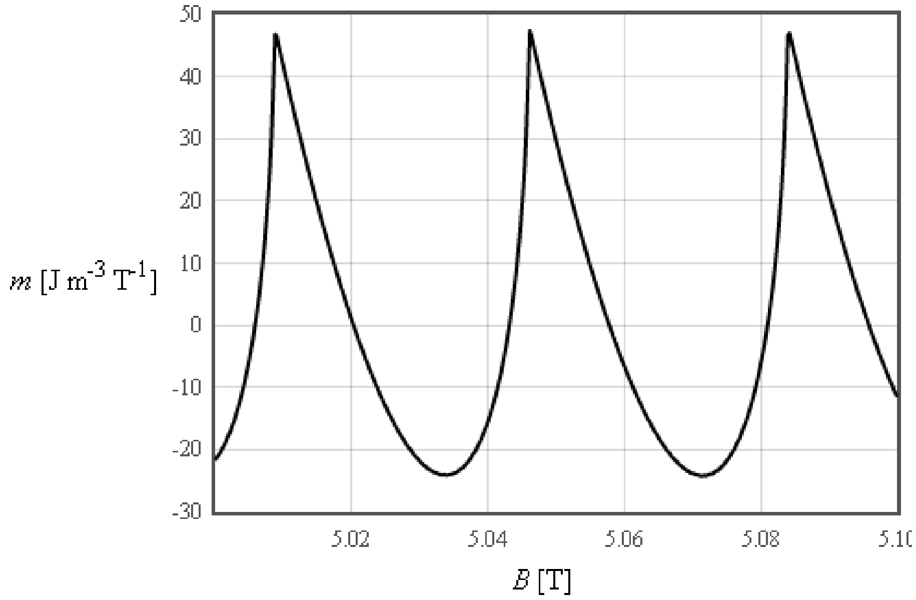
At $T = 0$

$$u = \frac{(2m)^{3/2}\omega_c}{4\pi^2\hbar^2} \sum_{\nu=0}^{\nu<\frac{E_F}{\hbar\omega_c}-\frac{1}{2}} \int_{\hbar\omega_c(\nu+\frac{1}{2})}^{E_F} \frac{E dE}{\sqrt{E-\hbar\omega_c(\nu+\frac{1}{2})}} \quad \text{J m}^{-3}$$

$$u = \frac{(2m)^{3/2}\omega_c}{6\pi^2\hbar^2} \sum_{\nu=0}^{\nu<\frac{E_F}{\hbar\omega_c}-\frac{1}{2}} \left(2\hbar\omega_c(\nu+\tfrac{1}{2})+E_F\right)\sqrt{E_F-\hbar\omega_c(\nu+\tfrac{1}{2})} \quad \text{J m}^{-3}$$

Magnetization 3d

$$m = -\frac{du}{dB}$$



Periodic in $1/B$

At finite temperatures this function would be smoother

de Haas - van Alphen oscillations

Practically all properties are periodic in 1/*B*

Internal energy

$$u = \int_{-\infty}^{\infty} E D(E) f(E) dE$$

Specific heat

$$c_v = \left(\frac{\partial u}{\partial T}\right)_{V=const}$$

Entropy

$$s = \int \frac{c_v}{T} dT$$

Helmholtz free energy

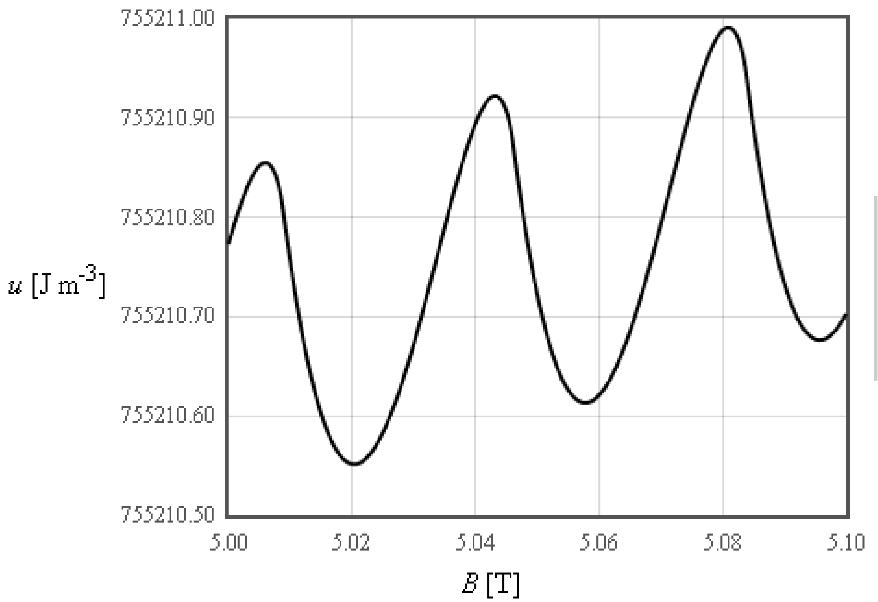
$$f = u - Ts$$

Pressure

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T=const}$$

Bulk modulus

$$B = -V \frac{\partial P}{\partial V}$$



Magnetization

$$M = -\frac{dU}{dH}$$