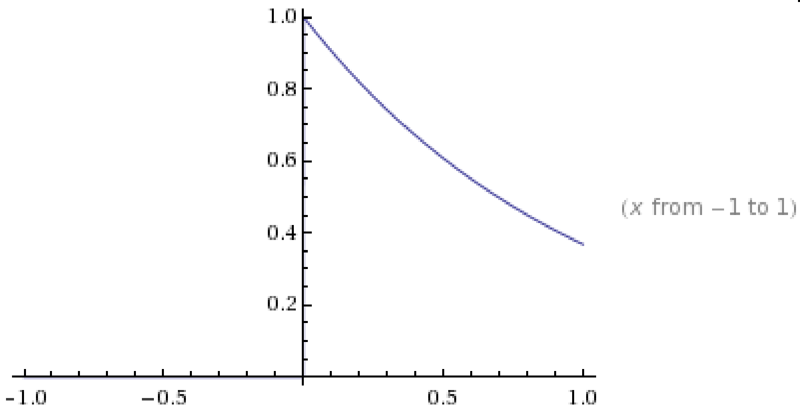
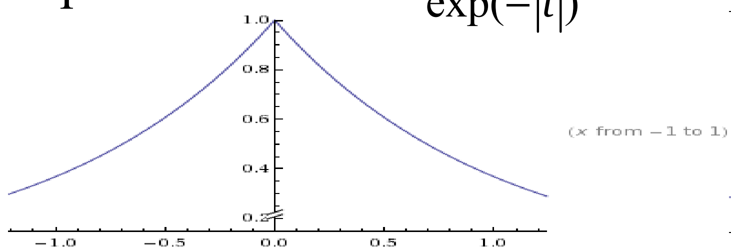
Linear response theory

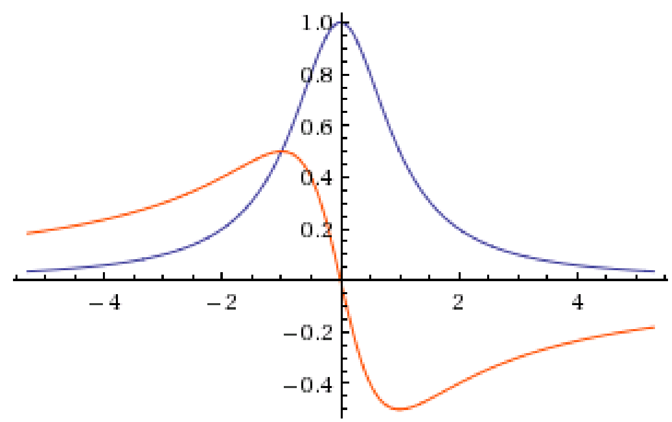


$$\chi(\omega) = \frac{1}{m}\frac{\frac{b}{m} - i\omega}{\left(\frac{b}{m}\right)^2 + \omega^2}$$

even component

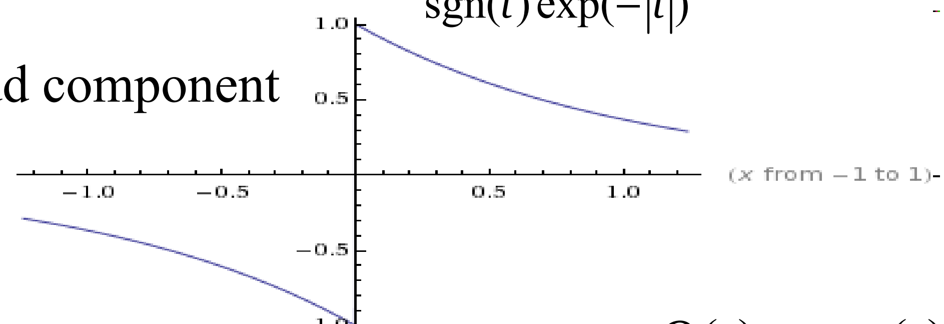
$\exp(-|t|)$





$\mathrm{sgn}(t)\exp(-|t|)$

odd component



$$O(t) = \mathrm{sgn}(t)E(t)$$

$$E(t) = \mathrm{sgn}(t)O(t)$$

Causality and the Kramers-Kronig relations (I)

$$\chi(\omega) = \int g(\tau) e^{-i\omega\tau} d\tau = \int E(\tau)\cos(\omega\tau)d\tau - i\int O(\tau)\sin(\omega\tau)d\tau = \chi'(\omega) + i\chi''(\omega)$$

The real and imaginary parts of the susceptibility are related.

If you know χ', inverse Fourier transform to find $E(t)$. Knowing $E(t)$ you can determine $O(t) = \text{sgn}(t)E(t)$. Fourier transform $O(t)$ to find χ".

$$\chi'(\omega) = \int_{-\infty}^{\infty} E(t)\cos(\omega t)dt \qquad E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi'(\omega)\cos(\omega t)d\omega$$

$$\boxed{O(t) = \text{sgn}(t)E(t) \qquad E(t) = \text{sgn}(t)O(t)}$$

$$\chi''(\omega) = -\int_{-\infty}^{\infty} O(t)\sin(\omega t)dt \qquad O(t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \chi''(\omega)\sin(\omega t)d\omega$$

Causality and the Kramers-Kronig relation (II)

Real space

$$E(t) = \mathrm{sgn}(t) O(t)$$

$$O(t) = \mathrm{sgn}(t) E(t)$$

Reciprocal space

$$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

$$\chi''(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$$

$$\chi' = \frac{-i}{\pi\omega} * i\chi'', \quad i\chi'' = \frac{-i}{\pi\omega} * \chi'$$

Take the Fourier transform, use the convolution theorem.

P: Cauchy principle value (go around the singularity and take the limit as you pass by arbitrarily close)

Singularity makes a numerical evaluation more difficult.

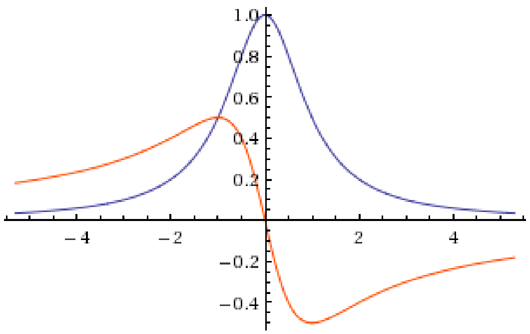
http://lamp.tu-graz.ac.at/~hadley/ss2/linearresponse/causality.php

Kramers-Kronig relations (III)

$$\chi''(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$$

$$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

Kramers-Kronig relations II



$$\chi'(\omega) = \chi'(-\omega)$$

$$\chi''(\omega) = -\chi''(-\omega)$$

Real part is even
Imaginary part is odd

$$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{0} \frac{\chi''(\omega')}{\omega' - \omega} d\omega' - \frac{1}{\pi} P \int_{0}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

change variables ω'→-ω'
(4 minus signs)

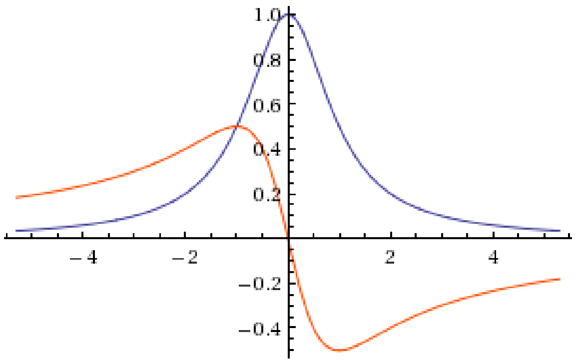
Kramers-Kronig relations (III)

$$\chi'(\omega) = -\frac{1}{\pi} P\int_0^{\infty} \frac{\chi''(\omega')}{\omega' + \omega} d\omega' - \frac{1}{\pi} P\int_0^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

$$\frac{1}{\omega' + \omega} + \frac{1}{\omega' - \omega} = \frac{2\omega'}{(\omega')^2 - \omega^2}$$

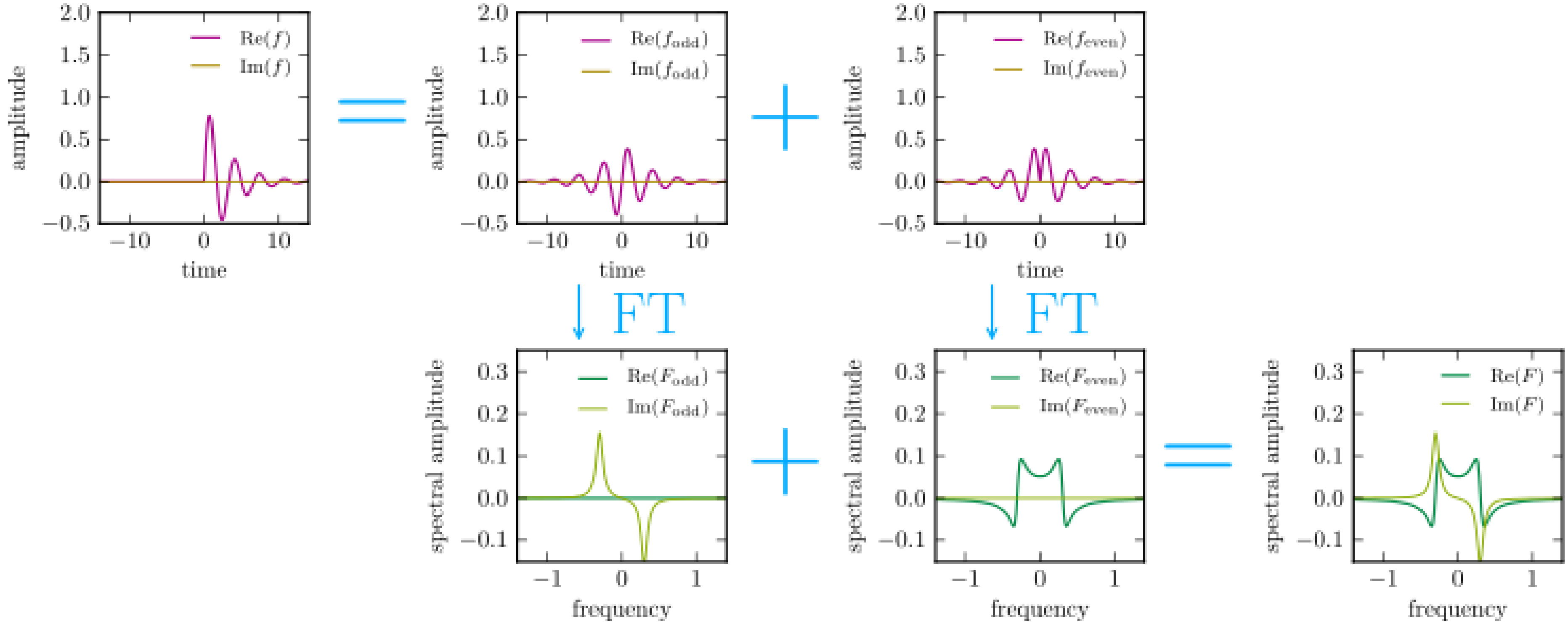
$$\chi'(\omega) = \frac{2}{\pi} P\int_0^{\infty} \frac{\omega'\chi''(\omega')}{(\omega')^2 - \omega^2} d\omega'$$

$$\chi''(\omega) = -\frac{2}{\pi} P\int_0^{\infty} \frac{\omega\chi'(\omega')}{(\omega')^2 - \omega^2} d\omega'$$



Singularity is stronger in this form.

Kramers-Kronig relations



If you know any of these for just positive frequencies,
you can calculate all the others.

https://en.wikipedia.org/wiki/Kramers%E2%80%93Kronig_relations

Impulse response/generalized susceptibility

The impulse response function is the response of the system to a δ-function excitation. The response function must be zero before the excitation.

The generalized susceptibility is the Fourier transform of the impulse response function.

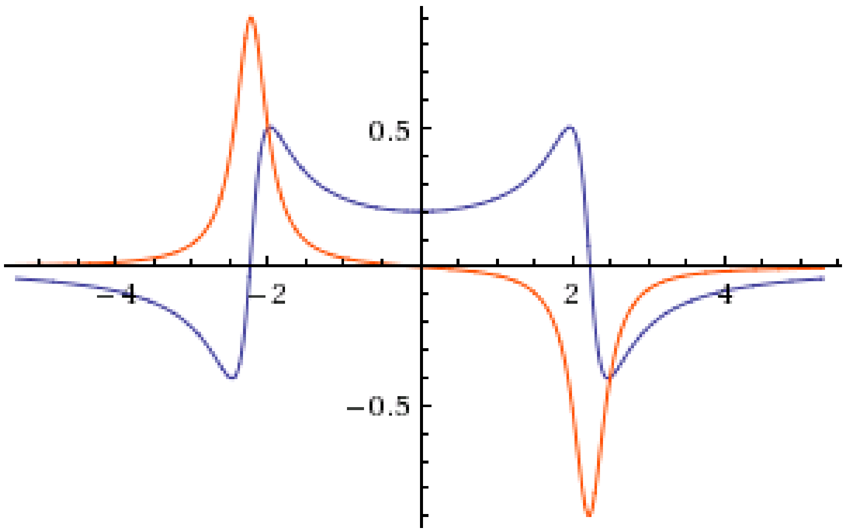
Any function that is zero before the excitation and nonzero afterwards must have both an odd component and an even component.

The generalized susceptibility must have a real and imaginary part. All information about the real part is contained in the imaginary part and vice versa.

Fluctuation-dissipation theorem

The fluctuation-dissipation theorem relates the size of the fluctuations to the dissipation in a system.

Most of the dissipation in a resonant system occurs at frequencies near the resonance.



http://en.wikipedia.org/wiki/Fluctuation_dissipation_theorem

Fluctuation-dissipation theorem

Brownian motion: The response to thermal noise is related to the viscosity.

$$m\frac{dv}{dt} = -\mu v \qquad\qquad D = \mu k_B T$$

Johnson noise: The voltage fluctuations are related to the resistance.

$$V_{rms} = \sqrt{4k_B TRB}$$

The fluctuation-dissipation theorem holds at equilibrium (where the equations are linear to a good approximation).

http://en.wikipedia.org/wiki/Fluctuation_dissipation_theorem

Dielectric response of insulators

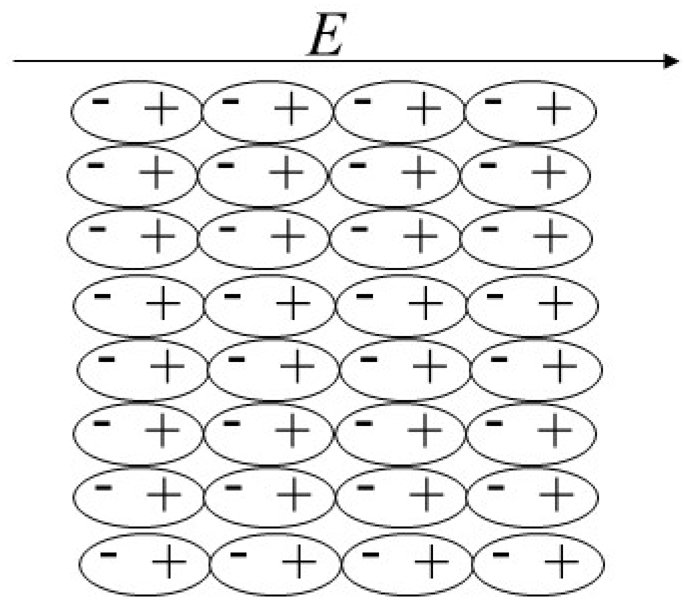
The electric polarization is related to the electric field

$$P_i = \varepsilon_0 \chi_{ij} E_j$$

The electric displacement vector D is also related to the electric field

$$D_i = P_i + \varepsilon_0 E_i = \varepsilon_0 (1 + \chi_{ij}) E_j = \varepsilon_0 \varepsilon_{ij} E_j$$

$$\varepsilon_{ij} = (1 + \chi_{ij})$$



E is decreased by a factor of the dielectric constant

Dielectric response of insulators

In an insulator, charge is bound. The response to an electric field can be modeled as a collection of damped harmonic oscillators

$$P = nex$$

Macroscopic polarization → P

density → n

ex = dipole moment

The core electrons of a metal respond to an electric field like this too.

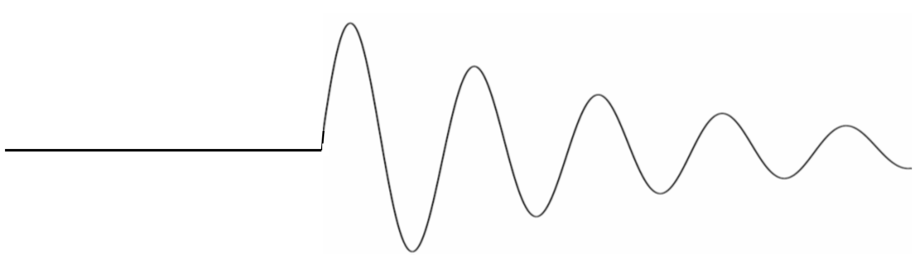
Dielectric response of insulators

The differential equation that describes how the position of the charge changes in time is:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=-eE(t)$$

The impulse response function is:

$$g(t)=-\frac{1}{b}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)\qquad t>0$$



Electric susceptibility

$$\vec{P} = \varepsilon_0 \chi_E \vec{E} \qquad \vec{P} = nq\vec{x}$$

$$\chi_E = \frac{P}{\varepsilon_0 E} = \frac{nqx}{\varepsilon_0 E}$$

Assume a solution of the form $x(\omega)e^{i\omega t}$, $E(\omega)e^{i\omega t}$

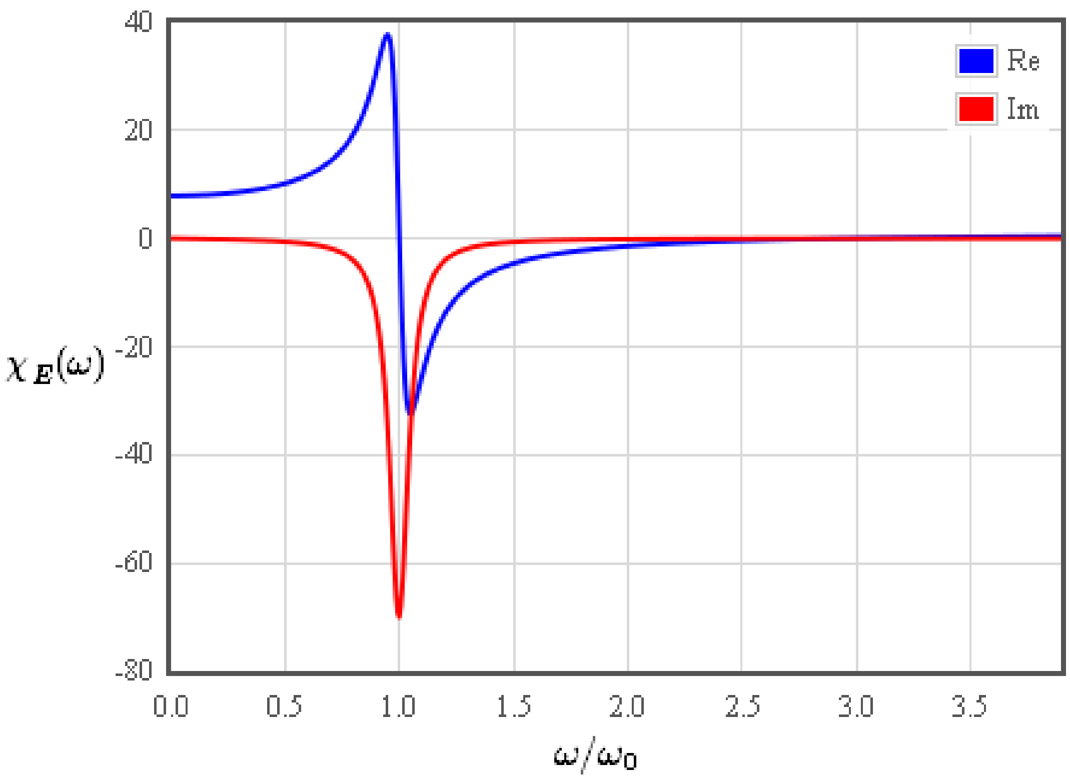
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = qE(t)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = -\frac{qE}{m}$$

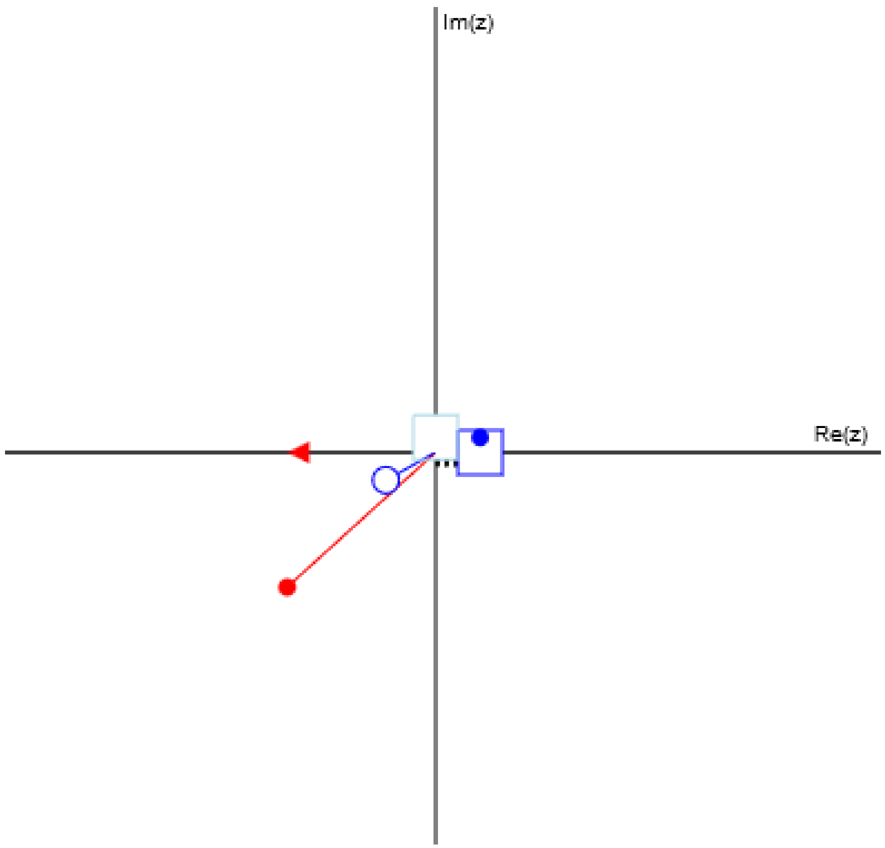
$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{m}$$

Electric susceptibility

$$\chi_E(\omega) = \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$



Resonance of a damped driven harmonic oscillator



$m = 4$ [kg]

$b = 1$ [N s/m]

$k = 6$ [N/m]

$F_0 = 0.9$ [N]

$\omega = 0.8$ [rad/s]

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 1.22 \text{ [rad/s]} = 0.194 \text{ [Hz]}$$

$$\theta = \text{atan}\left(\frac{\omega b}{k - m\omega^2}\right) = 0.228 \text{ [rad]} = 13.1 \text{ [deg]}$$

$$|A| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}} = 0.255 \text{ [m]}$$

$$Q = \frac{\sqrt{mk}}{b} = 4.90$$

Display $F_0 e^{i\omega t}$: ☑ Display $|A|e^{i(\omega t - \theta)}$: ☑

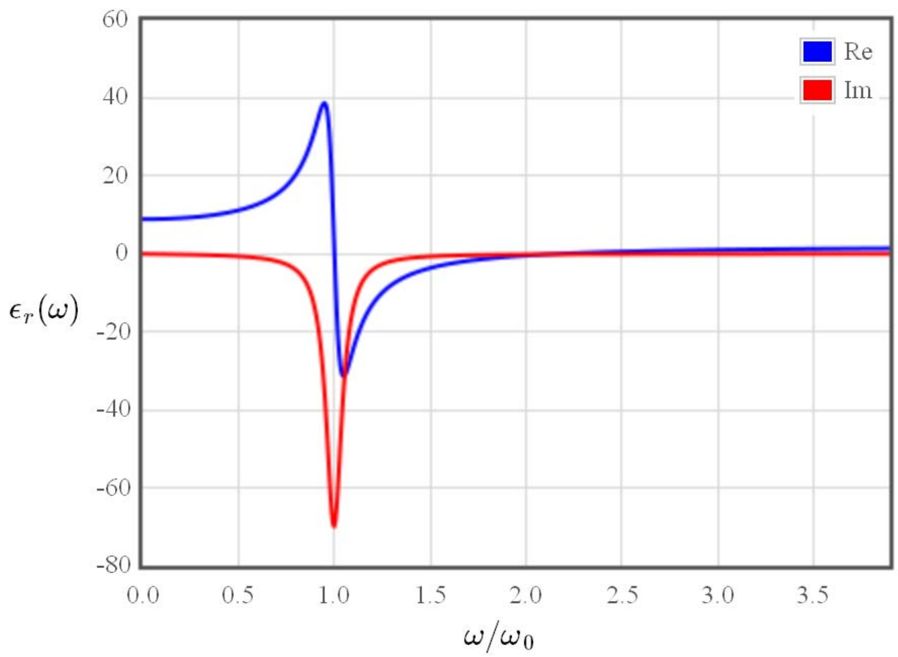
Display transients z: ☑ Display x_2: ☑

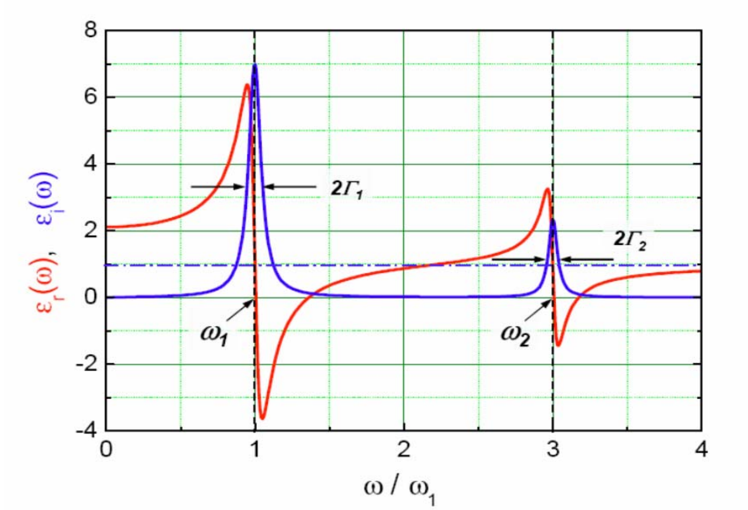
http://lamp.tu-graz.ac.at/~hadley/physikm/apps/resonance.en.php

Dielectric function

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \vec{P} + \epsilon_0 \vec{E}$$

$$\varepsilon_r(\omega) = 1 + \chi_E(\omega) = 1 + \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

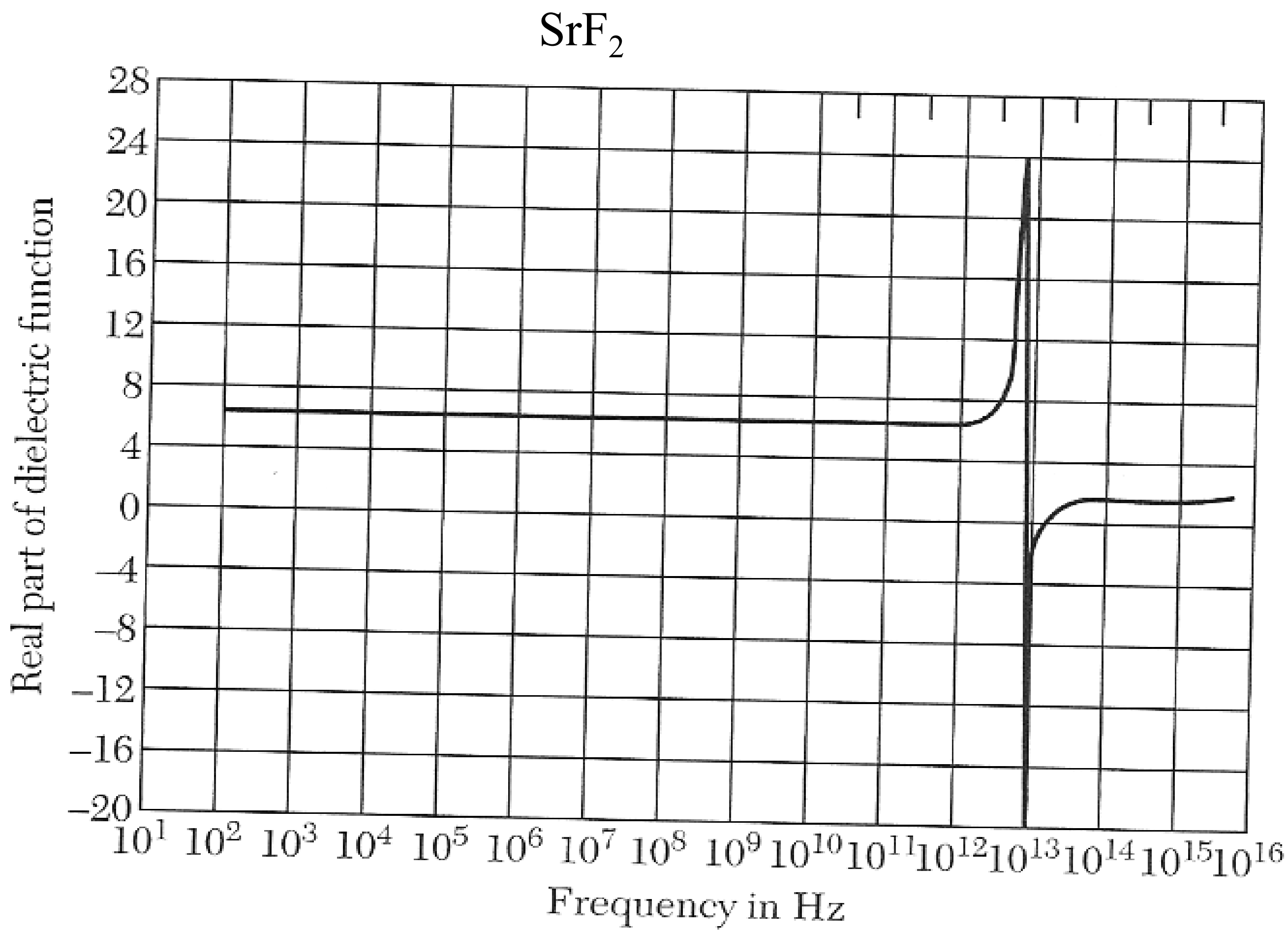




Gross and Marx

There can be more resonances.

Dielectric function of insulators



Insulators can often be modeled as a simple resonance.

Dispersion relation

In the section on photons, we derived the wave equation for light in vacuum. Here the wave equation for light in a dielectric material is derived.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

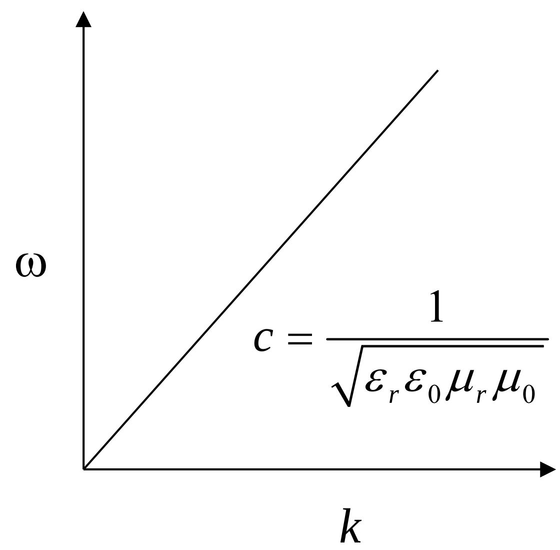
$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Take the curl $$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \nabla \times \vec{H}}{\partial t}$$

$$\nabla\left(\overset{0}{\nabla \cdot \vec{E}}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} = \nabla^2 \vec{D}$$



The normal mode solutions are plane waves: $\vec{D} = \vec{D}_0 \exp(\vec{k} \cdot \vec{r} - \omega t)$

$$\boxed{\varepsilon(\omega, k)\mu_0 \varepsilon_0 \omega^2 = k^2}$$

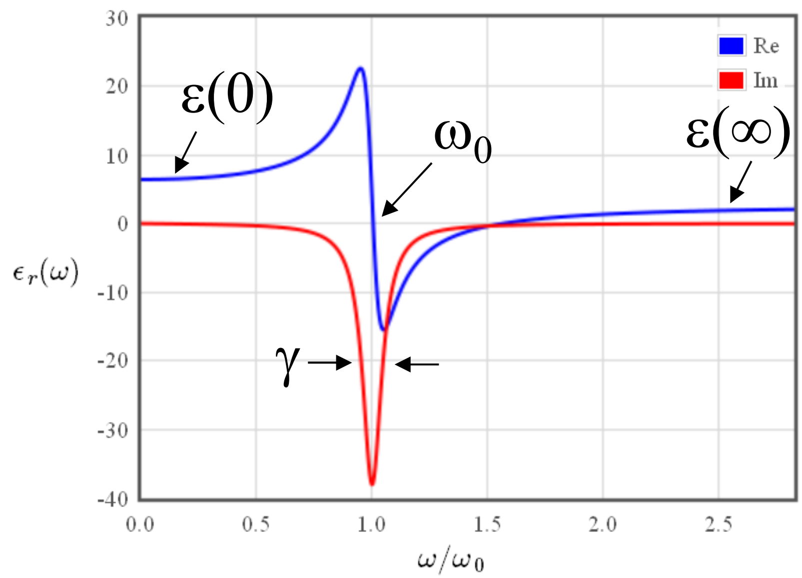
Dispersion relation

$$\varepsilon(\omega)\mu_0\varepsilon_0\omega^2 = k^2$$

If ε is real and positive: propagating electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)$

If $\varepsilon_r < 0$: decaying solutions $\exp(-\vec{k}\cdot\vec{r}-i\omega t)$

If ε is complex, $\varepsilon_r > 0$: decaying electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\exp(-\kappa r)$



Dielectric function

Dispersion relation: $\varepsilon_r \mu_0 \varepsilon_0 \omega^2 = k^2$

$$k = \sqrt{\varepsilon_r \mu_0 \varepsilon_0}\,\omega = \frac{\sqrt{\varepsilon_r}\,\omega}{c}$$

Measurable: $\sqrt{\varepsilon} = n + iK$

refractive index (n) — extinction coefficient (K)

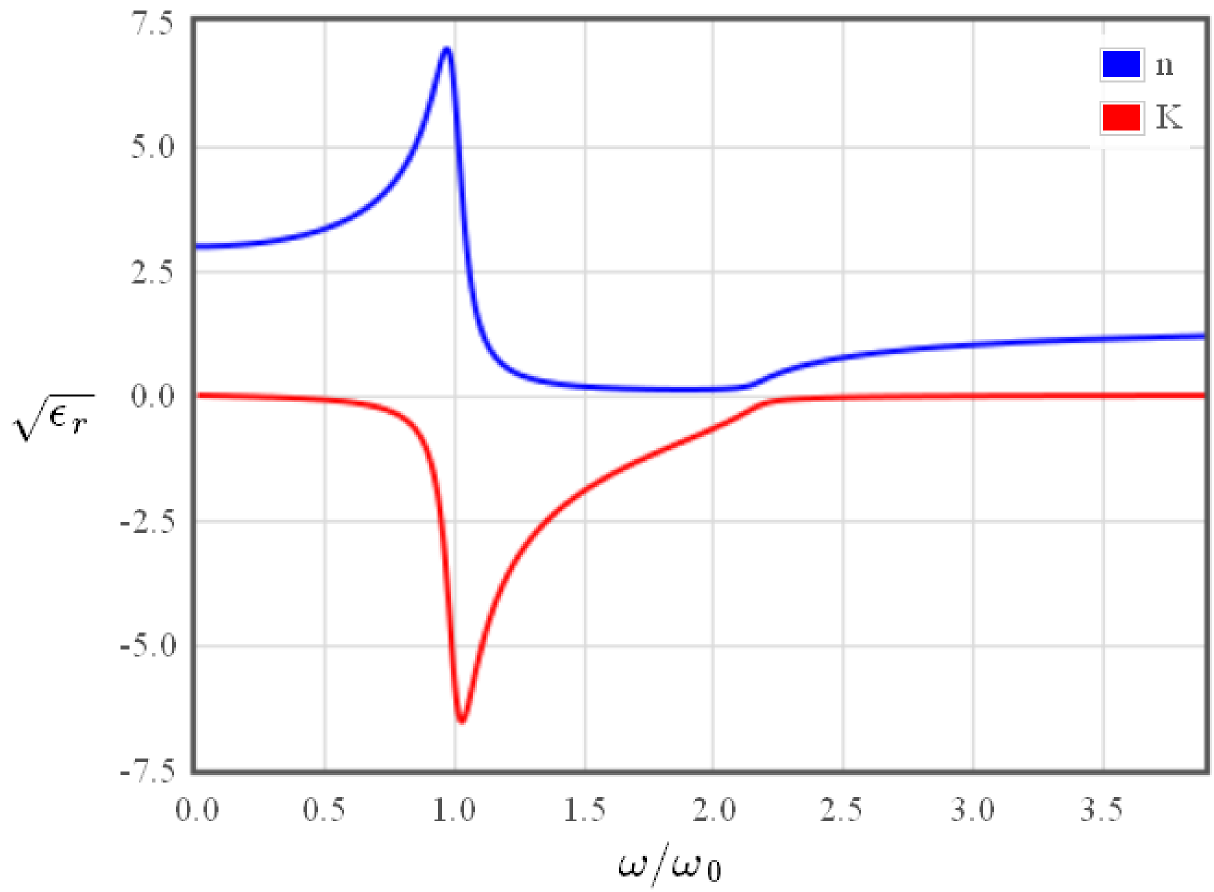
Vacuum	Dielectric
$\omega = ck$	$k = \frac{\sqrt{\varepsilon_r}\,\omega}{c}$
$\exp\left(i\left(\frac{\omega x}{c} - \omega t\right)\right)$	$\exp\left(i\left(\frac{(n+iK)\omega x}{c} - \omega t\right)\right)$
	$= \exp\left(i\left(\frac{n\omega x}{c} - \omega t\right)\right)\exp\left(-\frac{\omega K x}{c}\right)$

Intensity $I(x) = I(0)\exp(-\alpha x)$ J m^{-2} s^{-1} Beer-Lambert

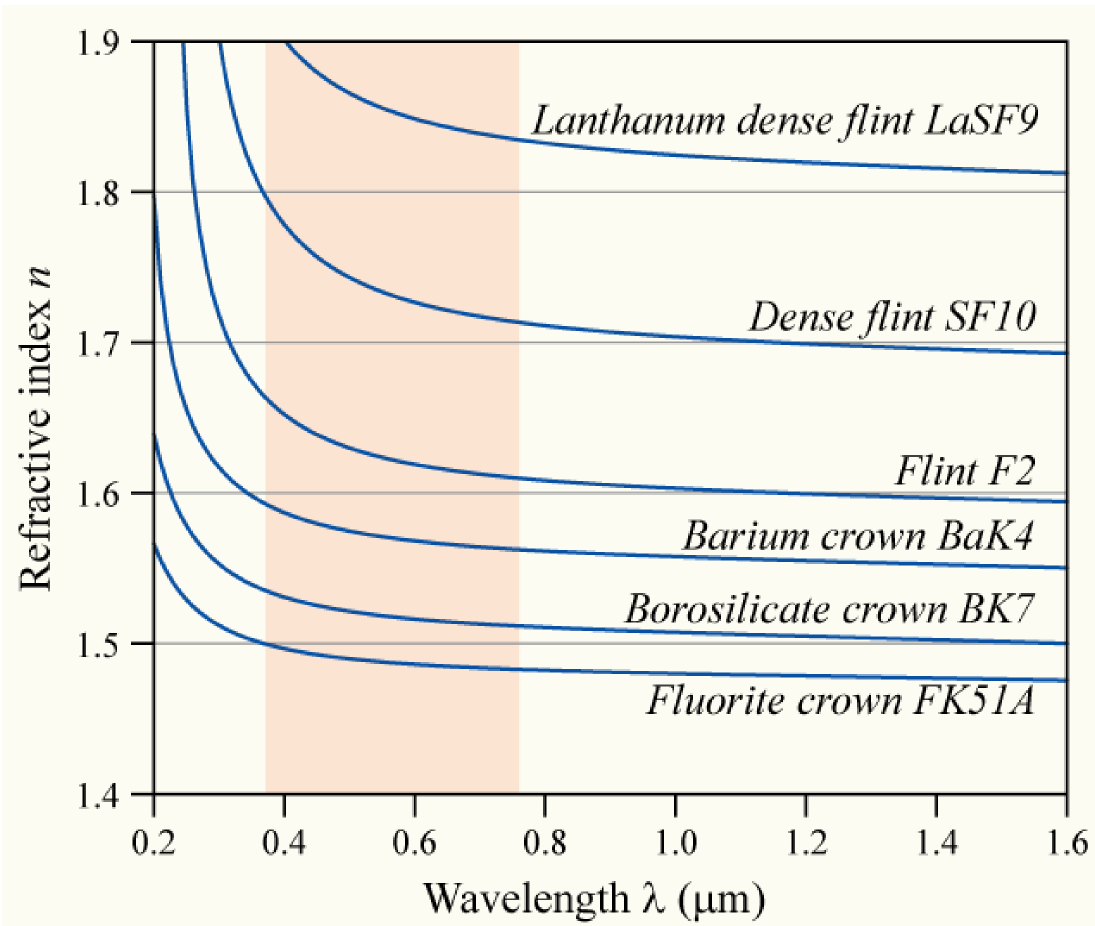
absorption coefficient $\longrightarrow$ $\alpha = \frac{2\omega K}{c}$

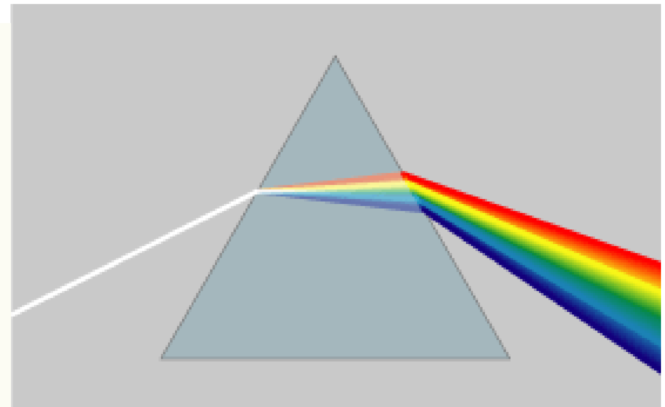
The index of refraction *n* and the extinction coefficient *K*

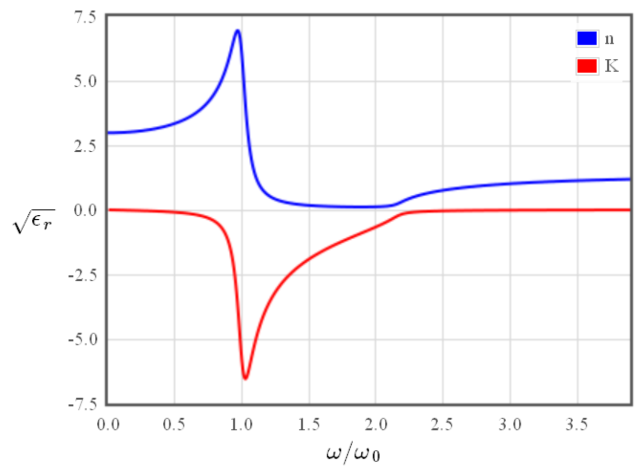
$$\sqrt{\epsilon_r} = n + iK$$



Dispersion







Cause of chromatic aberration in lenses.

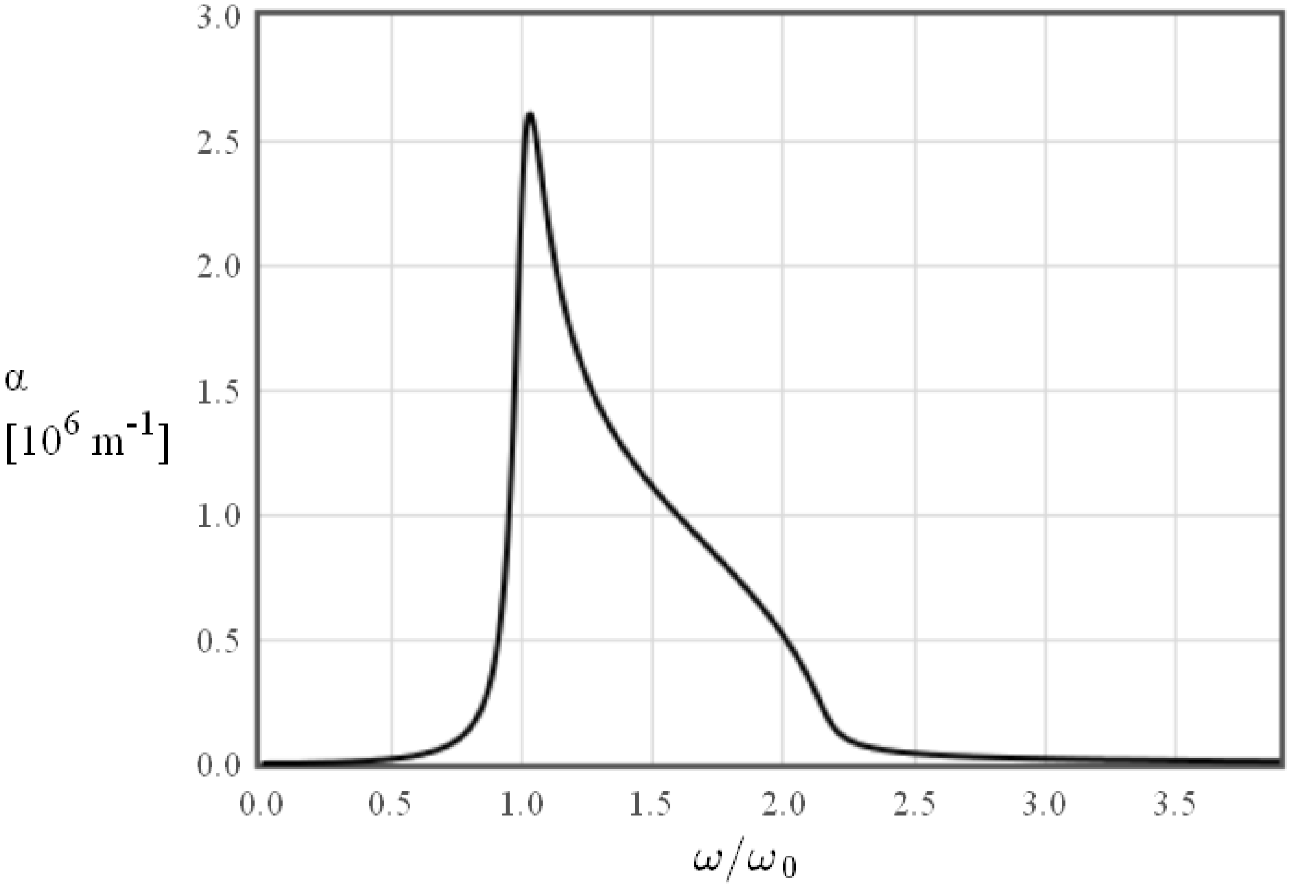
http://en.wikipedia.org/wiki/Dispersion_%28optics%29#mediaviewer/File:Prism_rainbow_schema.png

http://en.wikipedia.org/wiki/Refractive_index

Absorption coefficient α

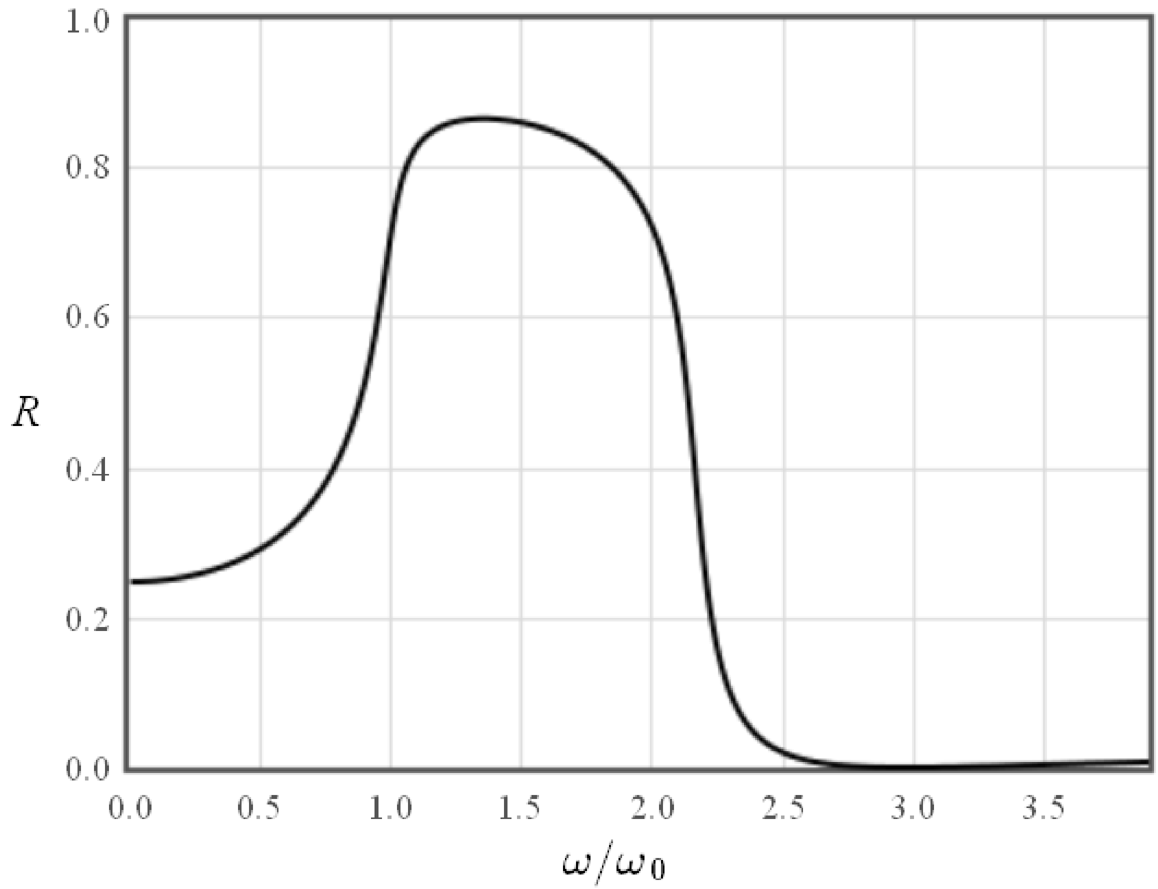
$$I = I_0 \exp(-\alpha x)$$

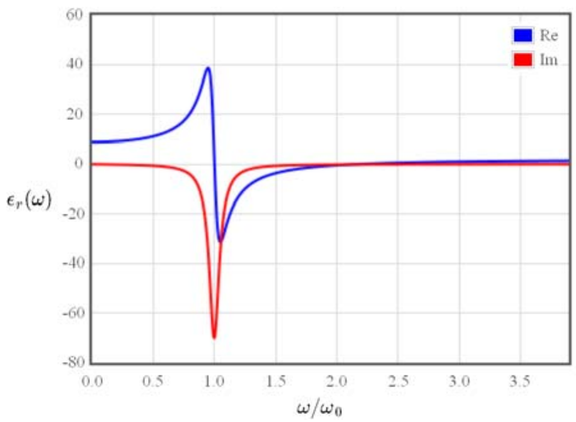
$$\alpha = \frac{2\omega K}{c}$$



Reflectance

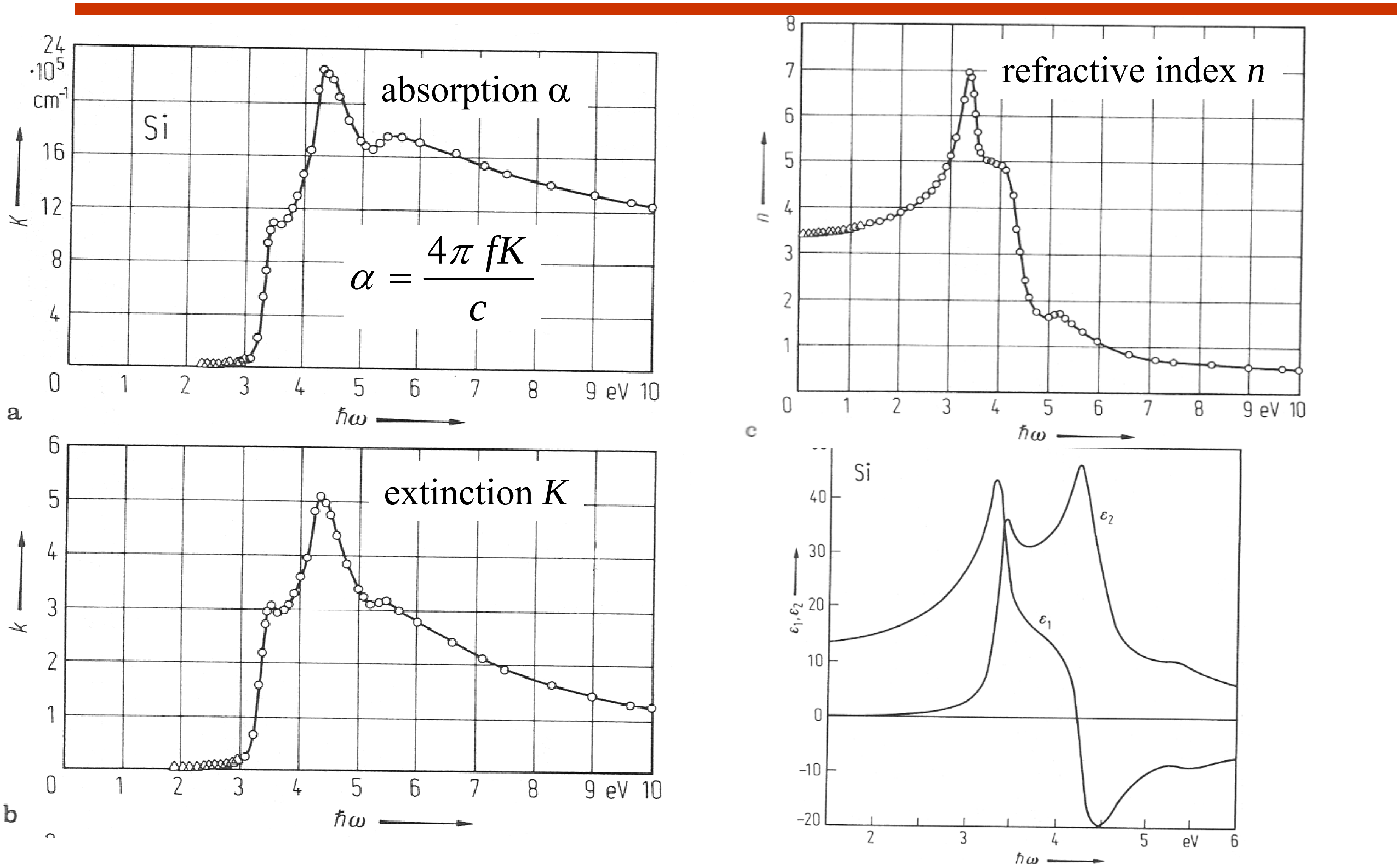
$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2}$$





Dielectric function of silicon

$$\sqrt{\varepsilon(\omega)} = n(\omega) + iK(\omega)$$



Optical properties of insulators and semiconductors

In an insulator, all charges are bound. By applying an electric field, the electrons and ions can be pulled out of their equilibrium positions. When this electric field is turned off, the charges oscillate as they return to their equilibrium positions. A simple model for an insulator can be constructed by describing the motion of the charge as a damped mass-spring system. The differential equation that describes the motion of a charge is,

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=-qE.$$

Rewriting above equation using $\omega_0=\sqrt{\frac{k}{m}}$ and the damping constant $\gamma=\frac{b}{m}$ yields,

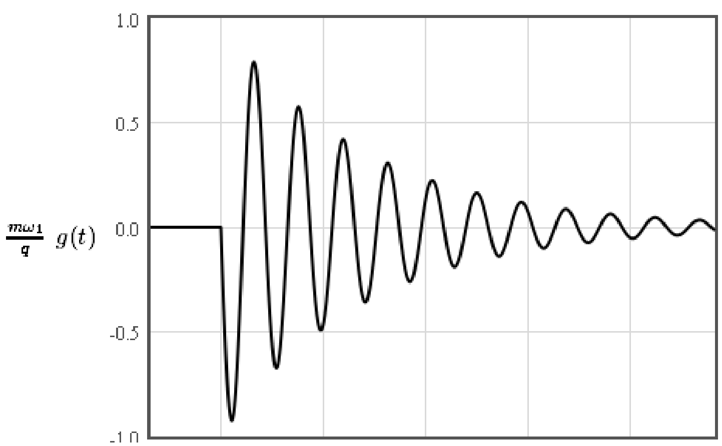
$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=-\frac{qE}{m}.$$

If the electric field is pulsed on, the response of the charges is described by the **impulse response function** $g(t)$. The impulse response function satisfies the equation,

$$\frac{d^2g}{dt^2}+\gamma\frac{dg}{dt}+\omega_0^2g=-\frac{q}{m}\delta(t).$$

The solution to this equation is zero before the electric field is pulsed on and at the time of the pulse the charges suddenly start oscillating with the frequency $\omega_1=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$. The amplitude of the oscillation decays exponentially to zero in a characteristic time $\frac{2}{\gamma}$.

$$g(t)=-\frac{q}{m\omega_1}\exp\left(-\frac{\gamma}{2}t\right)\sin(\omega_1 t).$$



Dielectrics

Dielectrics used as electrical insulators should not conduct.

Large breakdown field.

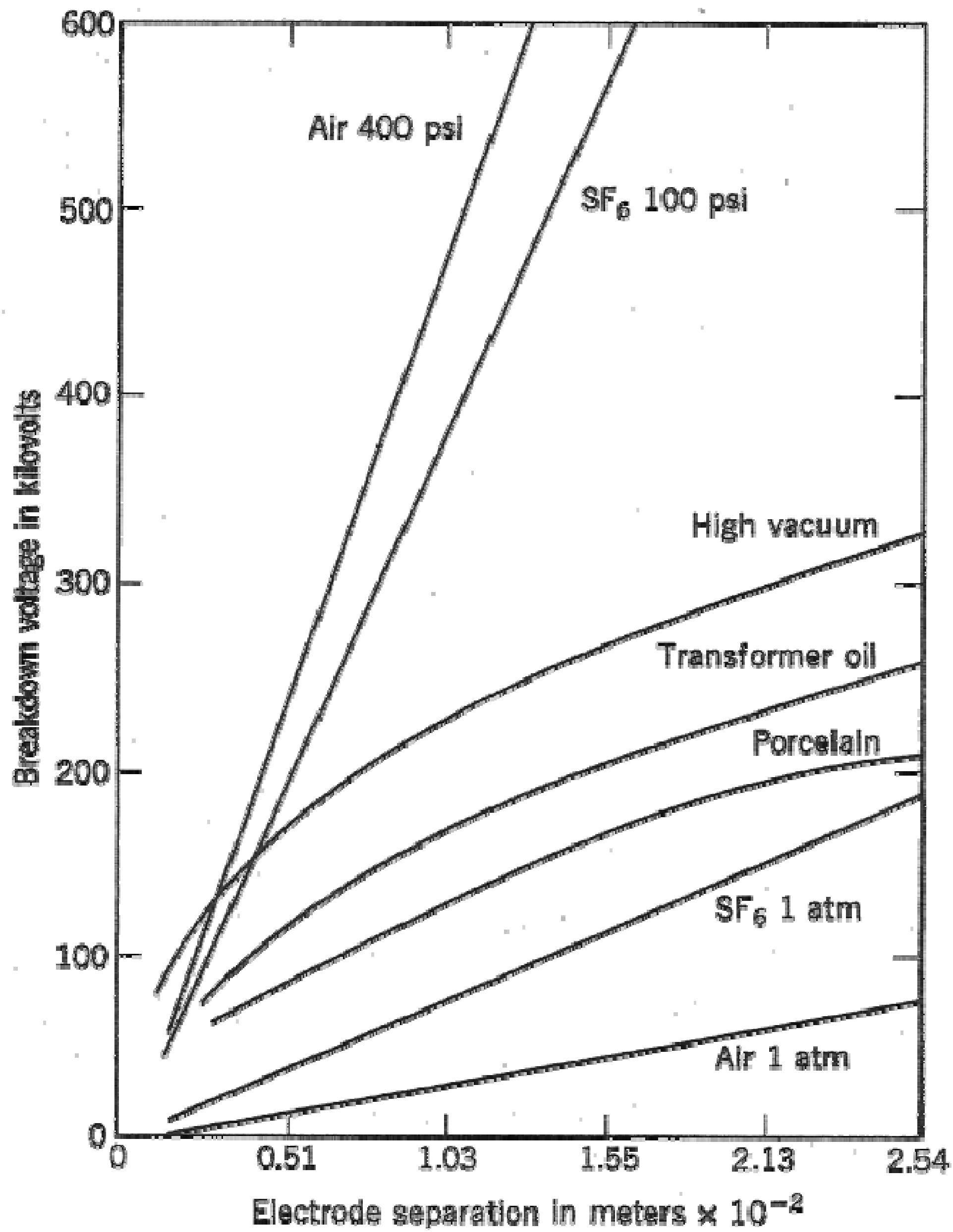
Low AC losses.

Sometimes a low dielectric constant is desired (CMOS interconnects)

Sometimes a high dielectric constant is desired (supercapacitors).

Breakdown field

Typically 10^5-10^6 V/cm



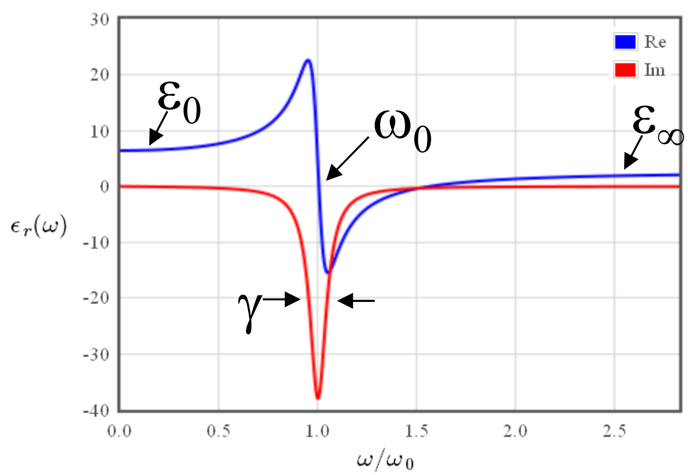
AC losses - loss tangent

In an ideal capacitor, current leads voltage by 90°.

Because the dielectric constant is complex, in real materials current leads voltage by 90° - δ.

$$\text{Power loss} = \frac{\omega \varepsilon_1 V_0^2}{2} \tan \delta$$

Becomes more of an issue at high frequencies (microwaves)



Loss tangent

Substance	Dielectric Constant (relative to air)	Dielectric Strength (V/mil)	Loss Tangent	Max Temp (°F)
ABS (plastic), Molded	2.0 - 3.5	400 - 1350	0.00500 - 0.0190	171 - 228
Air	1.00054	30 - 70		
Alumina - 96% - 99.5%	10.0 9.6		0.0002 @ 1 GHz 0.0002 @ 100 MHz 0.0003 @ 10 GHz	
Aluminum Silicate	5.3 - 5.5			
Bakelite	3.7			
Bakelite (mica filled)	4.7	325 - 375		
Balsa Wood	1.37 @ 1 MHz 1.22 @ 3 GHz		0.012 @ 1 MHz 0.100 @ 3 GHz	
Beeswax (yellow)	2.53 @ 1 MHz 2.39 @ 3 GHz		0.0092 @ 1 MHz 0.0075 @ 3 GHz	
Beryllium oxide	6.7		0.006 @ 10 GHz	
Butyl Rubber	2.35 @ 1 MHz 2.35 @ 3 GHz		0.001 @ 1 MHz 0.0009 @ 3 GHz	
Carbon Tetrachloride	2.17 @ 1 MHz 2.17 @ 3 GHz		<0.0004 @ 1 MHz 0.0004 @ 3 GHz	
Diamond	5.5 - 10			
Delrin (acetyl resin)	3.7	500		180
Douglas Fir	1.9 @ 1 MHz		0.023 @ 1 MHz	
Douglas Fir Plywood	1.93 @ 1 MHz 1.82 @ 3 GHz		0.026 @ 1 MHz 0.027 @ 3 GHz	
Enamel	5.1	450		
Epoxy glass PCB	5.2	700		
Ethyl Alcohol (absolute)	24.5 @ 1 MHz 6.5 @ 3 GHz		0.09 @ 1 MHz 0.25 @ 3 GHz	
Ethylene Glycol	41 @ 1 MHz 12 @ 3 GHz		-0.03 @ 1 MHz 1 @ 3 GHz	
Formica XX	4.00			
FR-4 (G-10) - low resin - high resin	4.9 4.2		0.008 @ 100 MHz 0.008 @ 3 GHz	
Fused quartz	3.8		0.0002 @ 100 MHz 0.00006 @ 3 GHz	
Fused silica (glass)	3.8			
Gallium Arsenide (GaAs)	13.1		0.0016 @ 10 GHz	
Germanium	16			
Glass	4 - 10			
Glass (Corning 7059)	5.75		0.0036 @ 10 GHz	
Gutta-percha	2.6			
Halowax oil	4.8			
High Density Polyethylene (HDPE), Molded	1.0 - 5.0	475 - 3810	0.0000400 - 0.00100	158 - 248
Ice (pure distilled water)	4.15 @ 1 MHz 3.2 @ 3 GHz		0.12 @ 1 MHz 0.0009 @ 3 GHz	
Kapton® Type 100 Type 150	3.9 2.9	7400 4400		500

http://www.rfcafe.com/references/electrical/dielectric-constants-strengths.htm

Polarizability

Overdamped modes

- Orientation polarizability
- Space charge polarizability

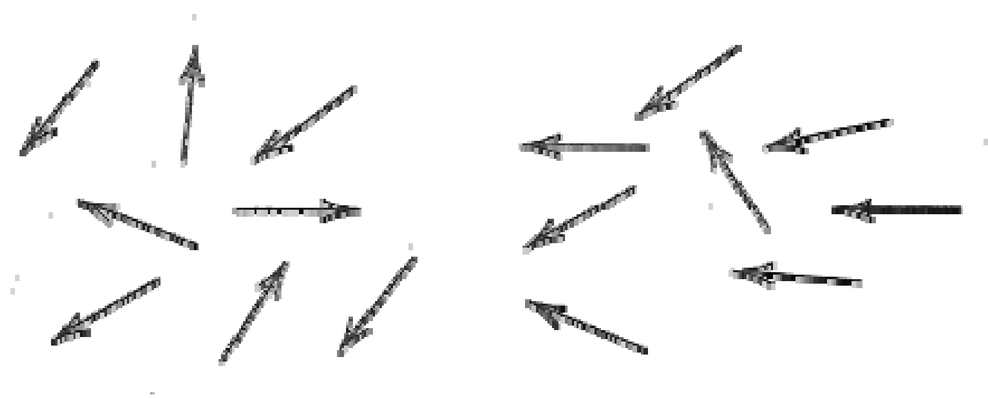
Underdamped modes

- Ionic polarizability
- Electronic polarizability

Orientation (dipolar) Polarizability

For materials (gases, liquids, solids) with a permanent dipole moment.

The theory is very similar to paramagnetism.



$$\chi \propto \frac{1}{T}$$ Curie law

Orientation Polarizability

Ion jumps.