The quantization of the electromagnetic field

Wave nature and the particle nature of light

Unification of the laws for electricity and magnetism (described by Maxwell's equations) and light

Quantization of the harmonic oscillator

Planck's radiation law

Serves as a template for the quantization of phonons, magnons, plasmons, electrons, spinons, holons and other quantum particles that inhabit solids.

Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In vacuum the source terms J and ρ are zero.

The vector potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Maxwell's equations in terms of A Coulomb gauge $\nabla \cdot \vec{A} = 0$

$$\nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0$$

$$\nabla \cdot \nabla \times \vec{A} = 0$$

$$\nabla \times \frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

The wave equation

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Using the identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\boxed{c^2 \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial t^2}.}$$

normal mode solutions have the form: $\vec{A}(\vec{r}, t) = \vec{A} \cos(\vec{k} \cdot \vec{r} - \omega t)$

Normal mode solutions

wave equation:

$$c^2\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial t^2}$$

normal mode solution:

$$\vec{A}(\vec{r},t) = \vec{A}\cos(\vec{k}\cdot\vec{r} - \omega t)$$

Normalschwingungen oder Normalmoden

put the solution into the wave equation

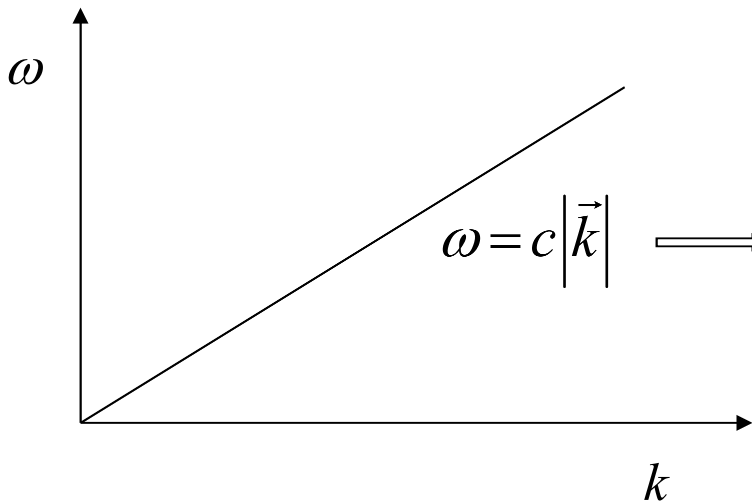
$$c^2k^2\vec{A} = \omega^2\vec{A}$$

dispersion relation

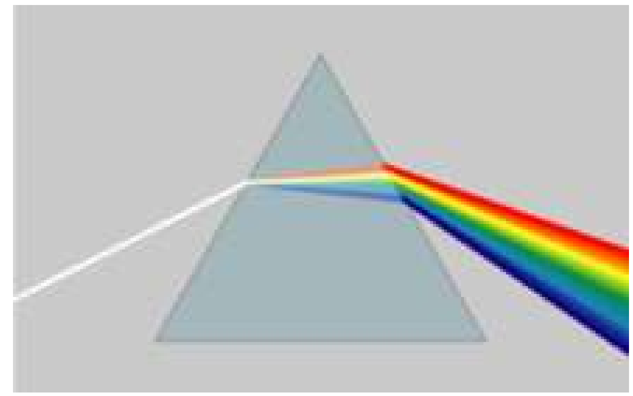
$$\omega = c\left|\vec{k}\right|$$

$$f = \frac{c}{\lambda}$$

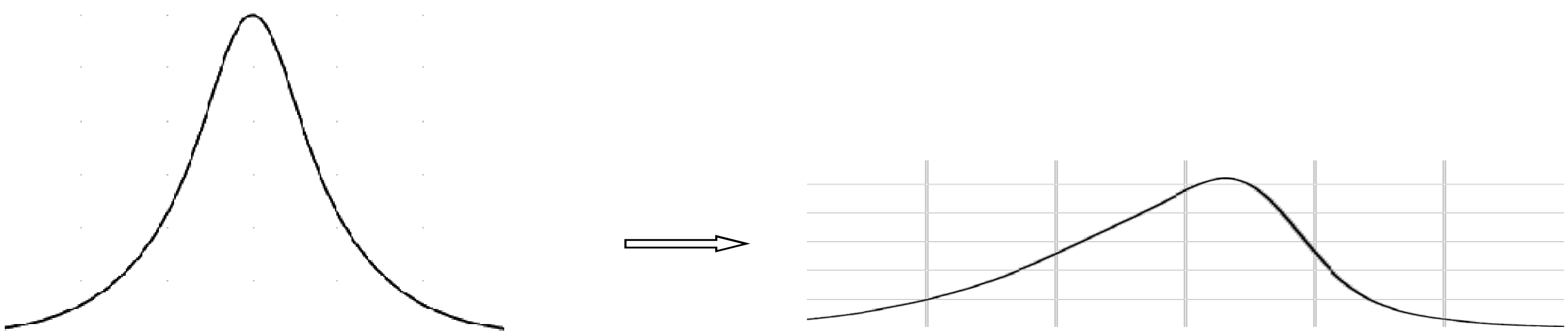
Dispersion relation



$$\omega = c\left|\vec{k}\right| \Longrightarrow f = \frac{c}{\lambda}$$



http://en.wikipedia.org/wiki/Dispersion_%28optics%29



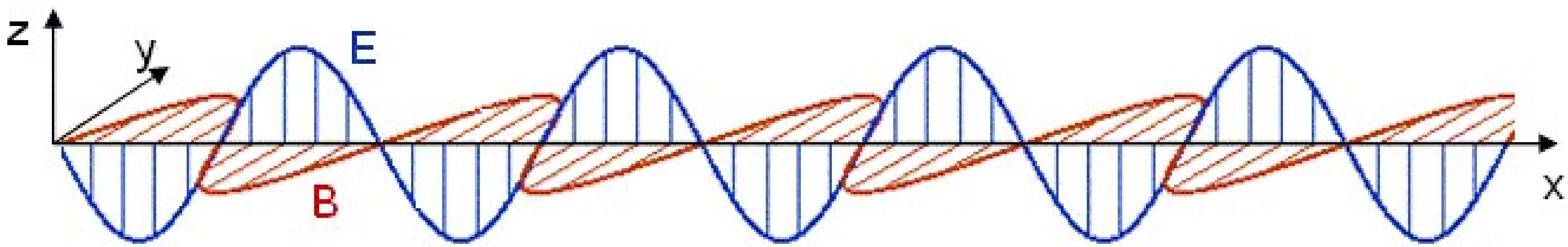
EM waves propagating in the *x* direction

$$\vec{A} = A_0 \cos(k_x x - \omega t)\hat{z}$$

The electric and magnetic fields are

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\omega A_0 \sin(k_x x - \omega t)\hat{z}$$

$$\vec{B} = \nabla \times \vec{A} = k_x A_0 \sin(k_x x - \omega t)\hat{y}$$



Quantization (using a trick)

The wave equation for a single mode.

$$-c^2\left(k_x^2+k_y^2+k_z^2\right)\vec{A}(\vec{k},t)=\frac{\partial^2\vec{A}(\vec{k},t)}{\partial t^2}$$

The equation for a single mode is mathematically equivalent to:

$$-\kappa x=m\frac{\partial^2 x}{\partial t^2}$$

$\kappa \leftrightarrow c^2k^2$, $m \leftrightarrow 1$

Quantization

Classical mathematical equivalence $\rightarrow$ quantum mathematical equivalence

$$E = \hbar\omega\left(j + \tfrac{1}{2}\right) \qquad j = 0, 1, 2\ldots$$

$$\omega = \sqrt{\frac{\kappa}{m}}$$

Rewriting this in terms of the electromagnetic field variables:

$\kappa \leftrightarrow c^2k^2$, $m \leftrightarrow 1$

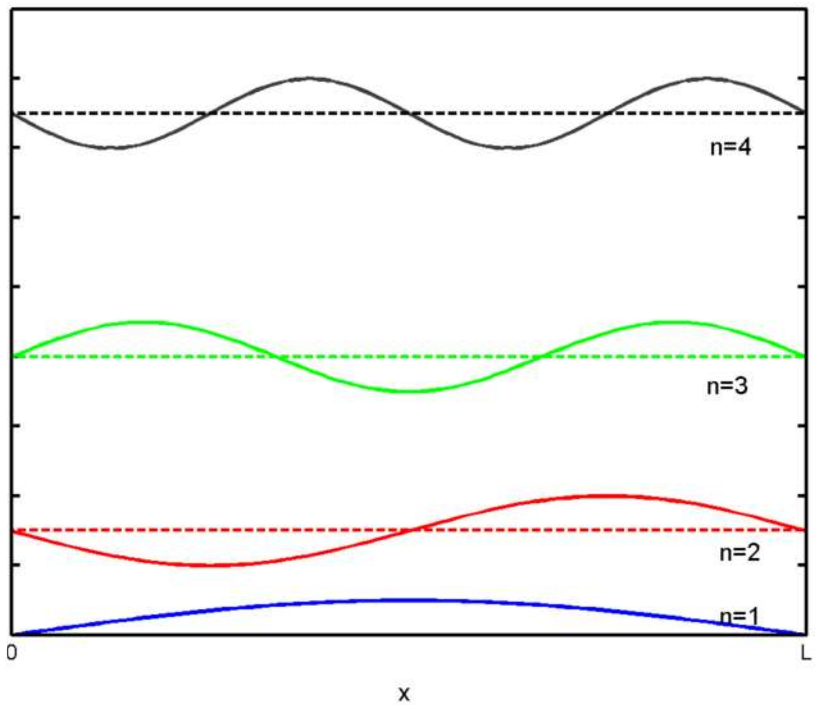
$$E = \hbar\omega\left(j + \tfrac{1}{2}\right) \qquad j = 0, 1, 2\ldots$$

Dispersion relation $\longrightarrow$ $\omega = c\left|\vec{k}\right|$

j is the number of photons in that mode

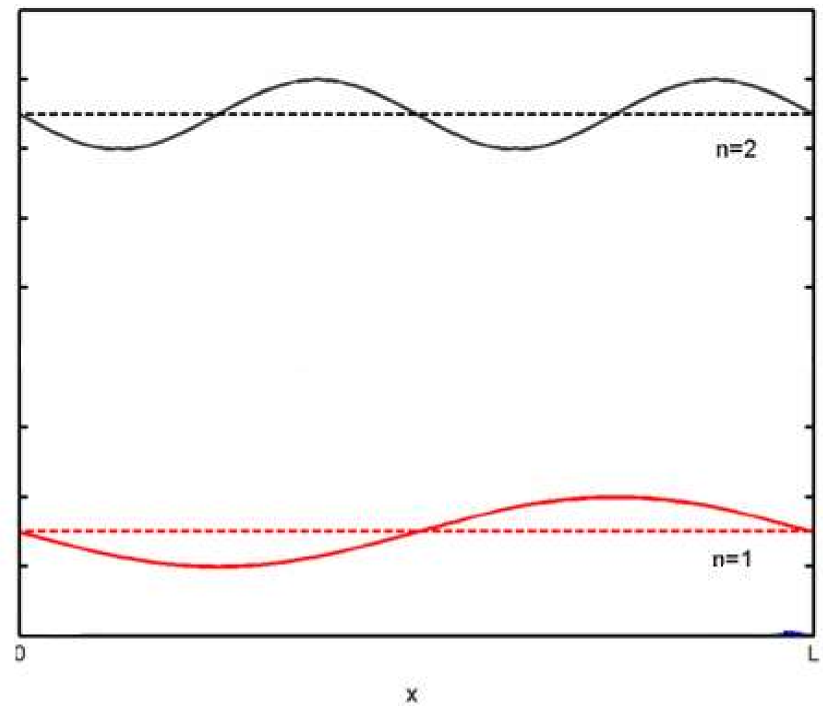
Boundary conditions

fixed boundary conditions



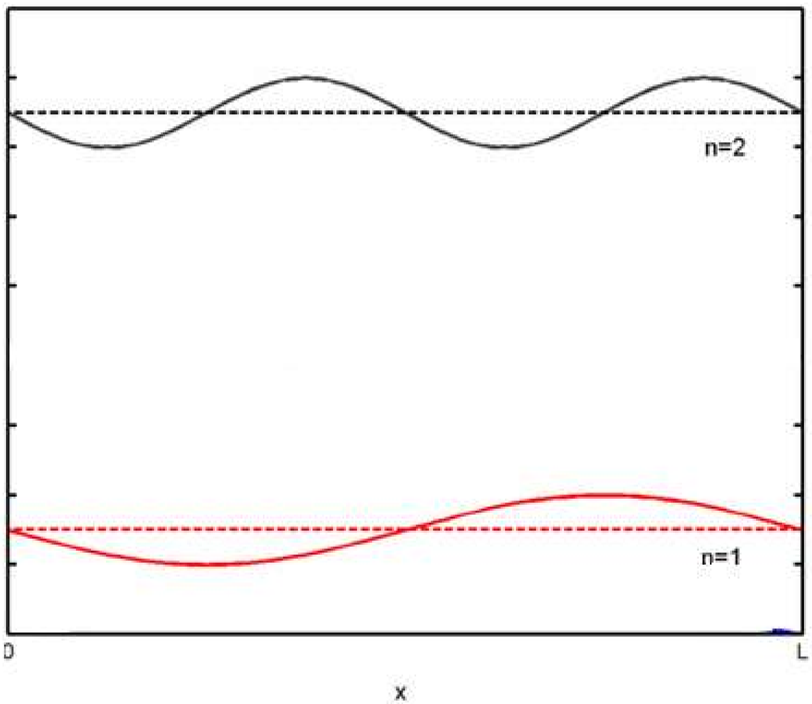
$$k = \frac{2\pi}{\lambda} = \frac{n\pi}{L}$$

periodic boundary conditions



$$k = \pm\frac{2\pi}{\lambda} = \pm\frac{2n\pi}{L}$$

Counting the normal modes

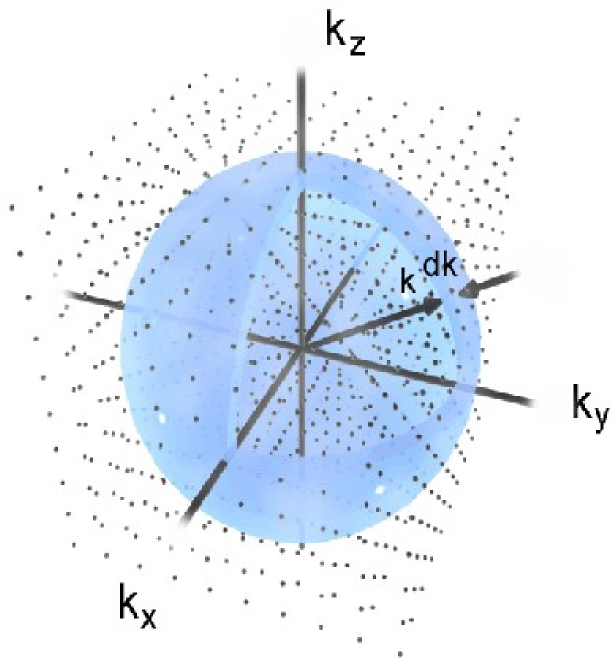


periodic boundary conditions

$$k = \pm\frac{2\pi}{\lambda} = \pm\frac{2n\pi}{L}$$

$$-\frac{8\pi}{L} \quad -\frac{6\pi}{L} \quad -\frac{4\pi}{L} \quad -\frac{2\pi}{L} \quad 0 \quad \frac{2\pi}{L} \quad \frac{4\pi}{L} \quad \frac{6\pi}{L} \quad \frac{8\pi}{L} \qquad k$$

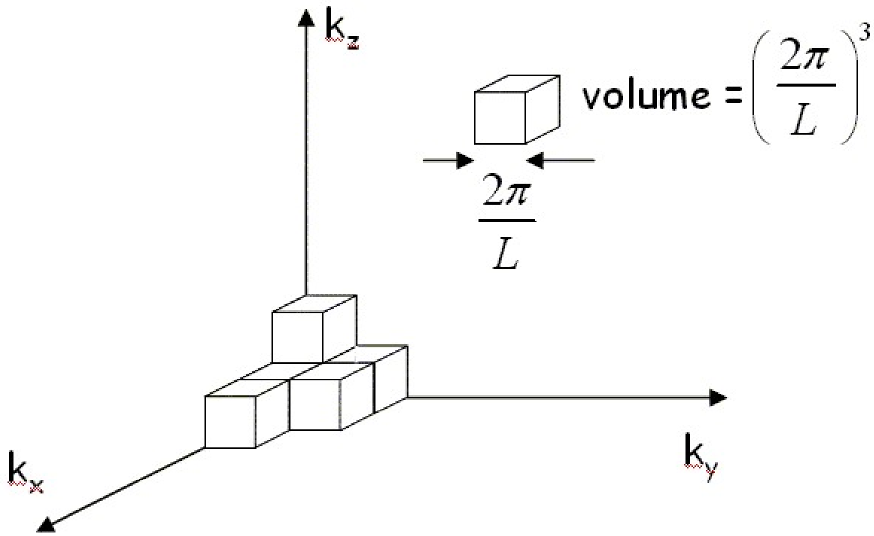
Density of states

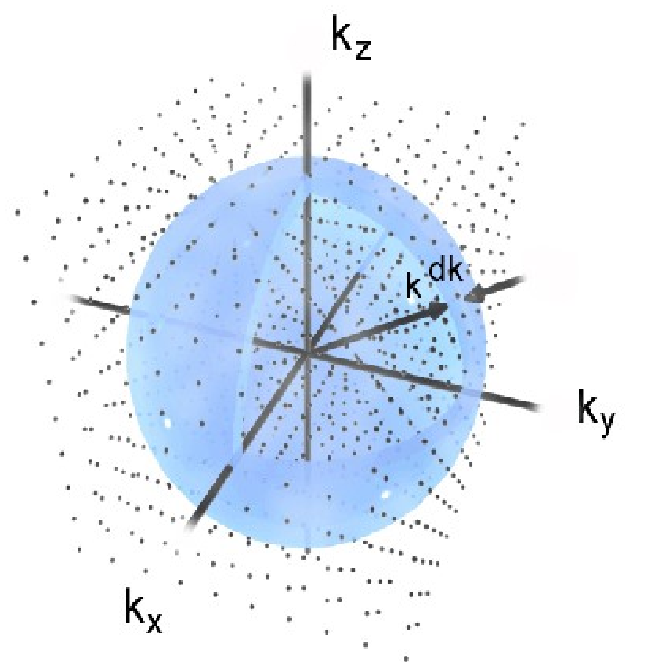


$$k_x, k_y, k_z = \ldots, \frac{-4\pi}{L}, \frac{-2\pi}{L}, 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \ldots$$

All states in the same shell have the same frequency.

Density of states





$$k_x, k_y, k_z = \ldots, \frac{-4\pi}{L}, \frac{-2\pi}{L}, 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \ldots$$

Number of states between k and $k+dk$ for a box of size L^3. =

$$L^3 D(k)dk = 2\frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} = \frac{k^2 L^3}{\pi^2}dk$$

polarizations

$D(k) = k^2/\pi^2$ = density of states/m^3

Density of states

The number of states per unit volume with a wavenumber between k and $k + dk$ is,

$$D(k)dk = \frac{k^2}{\pi^2}dk$$

$$\omega = ck \qquad \lambda = 2\pi/k$$

$$d\omega = c\,dk \qquad d\lambda = -2\pi/k^2 dk$$

The number of states per unit volume with a frequency between ω and $\omega + d\omega$ is,

$$D(\omega)d\omega = D(k)dk = \frac{\omega^2}{c^3\pi^2}d\omega.$$

The number of states per unit volume with a wavelength between λ and $\lambda + d\lambda$ is,

$$D(\lambda)d\lambda = D(k)dk = \frac{8\pi}{\lambda^4}d\lambda$$

Photons are Bosons

The mean number of bosons is given by the Bose-Einstein factor.

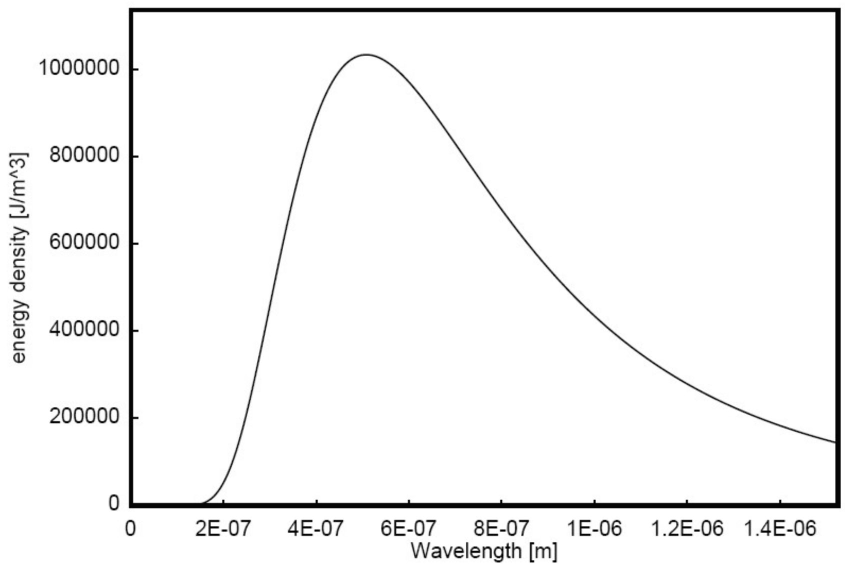
$$\frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}$$

Planck's radiation law

The energy density between λ and λ + *d*λ is the energy $E = hf = hc/\lambda$ of a mode times the density of modes, times the mean number of photons in that mode.

$$\underbrace{\frac{hc}{\lambda}}_{E} \cdot \underbrace{\frac{8\pi}{\lambda^4}}_{D(\lambda)} \cdot \underbrace{\frac{1}{e^{hc/\lambda k_B T}-1}}_{\text{Bose - Einstein factor}} d\lambda$$

$$u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T}-1}d\lambda \quad \mathrm{J/m^3}$$



Planck's radiation law, Wien's law

Planck's radiation law is often expressed in terms of the intensity

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} d\lambda \quad \text{W/m}^2$$

Differentiate to find the position of the peak

Wien's law: $\lambda_{max} T = 0.0028977$ m K

Stefan - Boltzmann law

Integrate intensity over all wavelengths

$$I = \int_0^\infty \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} d\lambda = \frac{2\pi^5 k_B^4 T^4}{15 h^3 c^2} = \sigma T^4 \quad \mathrm{W/m^2}$$

$$\sigma = 5.67051 \times 10^{-8} \qquad \mathrm{W\ m^{-2}\ K^{-4}}$$

Integrating the energy spectral density over all wavelengths

$$u = \frac{4\sigma T^4}{c} \quad \mathrm{J/m^3}$$

Thermodynamic quantities

Specific heat: $$c_v = \left(\frac{\partial u}{\partial T}\right)_V = \frac{16\sigma T^3}{c} \quad \text{J K}^{-1}\text{ m}^{-3}$$

entropy: $$s = \int \frac{c_v}{T} dT = \frac{16\sigma T^3}{3c} \quad \text{J K}^{-1}\text{ m}^{-3}$$

$$f = u - Ts$$

Helmholtz free energy: $$f = \frac{-4\sigma T^4}{3c} \quad \text{J/m}^3$$

Thermodynamic quantities

Radiation Pressure: $$P = -\frac{\partial F}{\partial V} = \frac{4\sigma V T^4}{3c} = \frac{4\sigma T^4}{3c} \quad \mathrm{N/m^2}$$

Momentum of a photon: $$\vec{p} = \hbar \vec{k}$$

Recipe for the quantization of fields

Determine the classical normal modes. If the equations are nonlinear, linearize the equations. The nonlinear terms can be included later as perturbations.

Calculate the density of states (density of normal modes per energy).

Quantize the states.

Knowing the distribution of the quantum states, deduce thermodynamic quantities.

Photons, phonons, magnons, plasmons, ...

We quantized the wave equation.

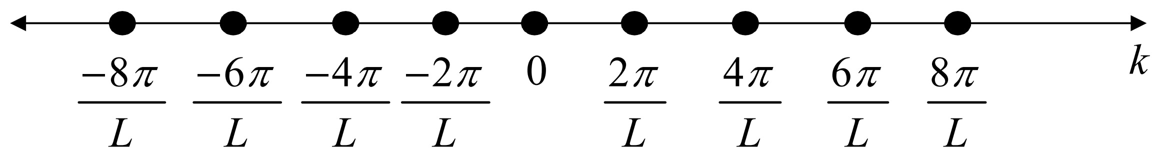
The wave equation describes the motion of light waves, sound waves, plasma waves, waves in the magnetization, waves in the electric polarization, ...

The density of states is different in 1 and 2 dimensions: waves on a string (carbon nanotubes), waves at a surface, waves at an interface.

Sound waves have 3 polarizations, light waves have 2.

Density of states

1-D



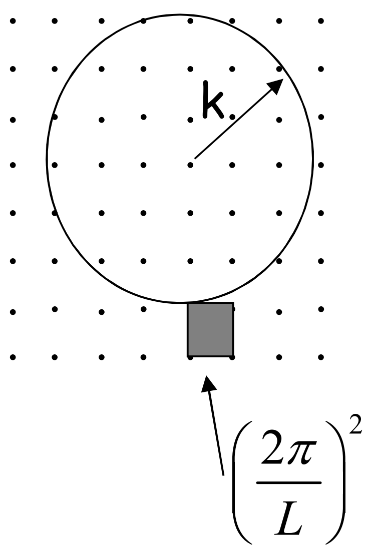
Number of states between $|k|$ and $|k|+dk$ for a line of size L. $= LD(k)dk = 2\cdot 2\cdot \dfrac{dk}{\frac{2\pi}{L}}$

polarizations

+/- k

$$D(k) = \frac{2}{\pi}$$

2-D

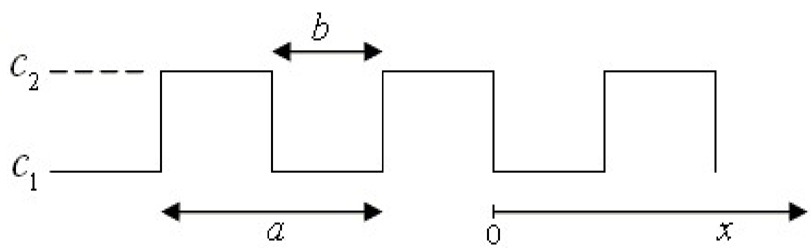


Number of states between $|k|$ and $|k|+dk$ for an area of size L^2. $= L^2 D(k)dk = 2\dfrac{2\pi k\ dk}{\left(\frac{2\pi}{L}\right)^2}$

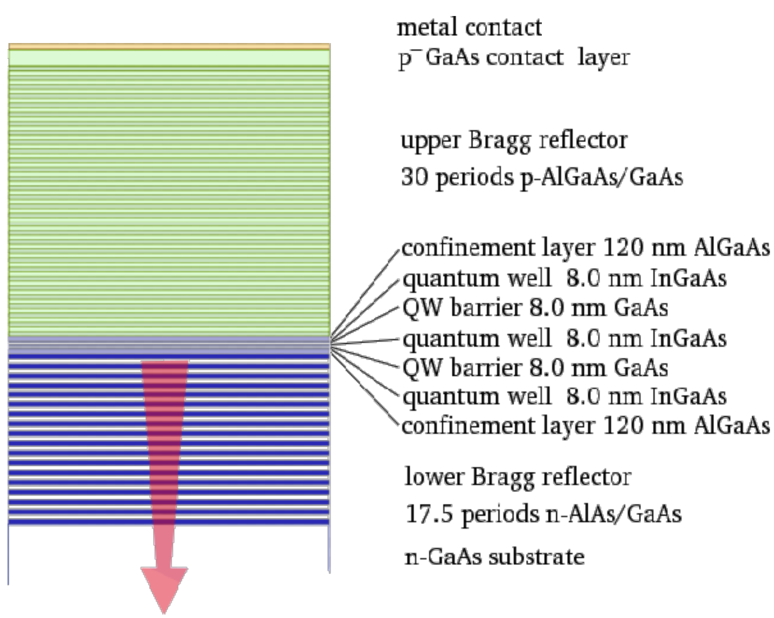
$$D(k) = \frac{k}{\pi} \qquad [\mathrm{m}^{-1}]$$

	1-D	2-D	3-D
Wave Equation c = speed of light $A_j = j^{th}$ component of the vector potential	$c^2 \frac{d^2 A_j}{dx^2} = \frac{d^2 A_j}{dt^2}$	$c^2 \left(\frac{d^2 A_j}{dx^2} + \frac{d^2 A_j}{dy^2} \right) = \frac{d^2 A_j}{dt^2}$	$c^2 \left(\frac{d^2 A_j}{dx^2} + \frac{d^2 A_j}{dy^2} + \frac{d^2 A_j}{dz^2} \right) = \frac{d^2 A_j}{dt^2}$
Eigenfunction solutions k = wavenumber ω = angular frequency	$A_j = \exp\left(i \left(kx - \omega t \right) \right)$	$A_j = \exp\left(i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right)$	$A_j = \exp\left(i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right)$
Dispersion relation	$\omega = ck$	$\omega = c\left\lvert \vec{k} \right\rvert$	$\omega = c\left\lvert \vec{k} \right\rvert$
Density of states	$D(k) = \frac{2}{\pi}$	$D(k) = \frac{k}{\pi}$ [m^{-1}]	$D(k) = \frac{k^2}{\pi^2}$ [m^{-2}]
Density of states $D(\omega) = D(k)\frac{dk}{d\omega}$	$D(\omega) = \frac{2}{\pi c}$ [s/m]	$D(\omega) = \frac{\omega}{\pi c^2}$ [s/m^2]	$D(\omega) = \frac{\omega^2}{\pi^2 c^3}$ [s/m^3]
Density of states $D(\lambda) = D(k)\frac{dk}{d\lambda}$ λ = wavelength	$D(\lambda) = \frac{4}{\lambda^2}$ [m^{-2}]	$D(\lambda) = \frac{4\pi}{\lambda^3}$ [m^{-3}]	$D(\lambda) = \frac{8\pi}{\lambda^4}$ [m^{-4}]
Density of states $D(E) = D(\omega)\frac{d\omega}{dE}$	$D(E) = \frac{2}{\pi \hbar c}$ [J^{-1}m^{-1}]	$D(E) = \frac{E}{\pi \hbar^2 c^2}$ [J^{-1}m^{-2}]	$D(E) = \frac{E^2}{\pi^2 \hbar^3 c^3}$ [J^{-1}m^{-3}]
Chemical potential	$\mu = 0$	$\mu = 0$	$\mu = 0$
Intensity spectral density $k_B = 1.3806504 \times 10^{-23}$ [J/K] Boltzmann's constant $h = 6.62606896 \times 10^{-34}$ [J s] Planck's constant	$I(\lambda) = \frac{2hc^2}{\lambda^3 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J m^{-1}s^{-1}]	$I(\lambda) = \frac{4hc^2}{\lambda^4 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J m^{-2}s^{-1}]	$I(\lambda) = \frac{2\pi hc^2}{\lambda^5 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J m^{-3}s^{-1}]
Wien's law $\left. \frac{dI(\lambda)}{d\lambda} \right\rvert_{\lambda = \lambda_{max}} = 0$	$\lambda_{max} = \frac{0.0050994367}{\mathrm{T}}$ [m]	$\lambda_{max} = \frac{0.0036696984}{\mathrm{T}}$ [m]	$\lambda_{max} = \frac{0.002897707138}{\mathrm{T}}$ [m]
Stefan - Boltzmann law $I = \int_0^\infty I(\lambda) d\lambda$ $\zeta(3) \approx 1.202$ Riemann ζ function $\sigma = 5.67 \times 10^{-8}$ Stefan-Boltzmann constant	$I = \frac{\pi^2 k_B{}^2 T^2}{3h}$ [J s^{-1}]	$I = \frac{8\varsigma(3) k_B{}^3 T^3}{h^2 c}$ [J m^{-1} s^{-1}]	$I = \frac{2\pi^5 k_B^4 T^4}{15 c^2 h^3} = \sigma T^4$ [J m^{-2} s^{-2}]
Internal energy distribution $u(\lambda) = \underbrace{\frac{hc}{\lambda}}_{E} \cdot \frac{D(\lambda)}{\exp\left(\frac{hc}{\lambda k_B T} \right) - 1}$	$u(\lambda) = \frac{4hc}{\lambda^3 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J/m^2]	$u(\lambda) = \frac{4\pi hc}{\lambda^4 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J/m^3]	$u(\lambda) = \frac{8\pi hc}{\lambda^5 \left(\exp\left(\frac{hc}{\lambda k_B T} \right) - 1 \right)}$ [J/m^4]
Internal energy $u = \int_0^\infty u(\lambda) d\lambda$	$u = \frac{2\pi^2 k_B{}^2 T^2}{3hc}$ [J/m]	$u = \frac{8\varsigma(3) \pi k_B{}^3 T^3}{h^2 c^2}$ [J/m^2]	$u = \frac{4\sigma T^4}{c}$ [J/m^3]

Light in a layered material



The dielectric constant and speed of light are different for the two layers.



Distributed Bragg reflector

Light in a layered material

Wave equation in a periodic medium $$c^2(x)\frac{\partial^2 A_j}{\partial x^2}=\frac{\partial^2 A_j}{\partial t^2}$$

Separation of variables $$A_j\left(x,t\right)=\xi(x)e^{-i\omega t}$$

Hill's equation $$\frac{d^2\xi(x)}{dx^2}=-\frac{\omega^2}{c^2(x)}\xi(x)$$

Normal modes don't have a clearly defined wavelength.

2nd order linear differential equation with periodic coefficients.
Mathematically equivalent to the time independent Schrödinger equation.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}=\left(E-V(x)\right)\psi(x)$$

Differential equations

The solutions to a linear differential equation with constant coefficients,

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=d,$$

have the form,

$$e^{\lambda x}.$$

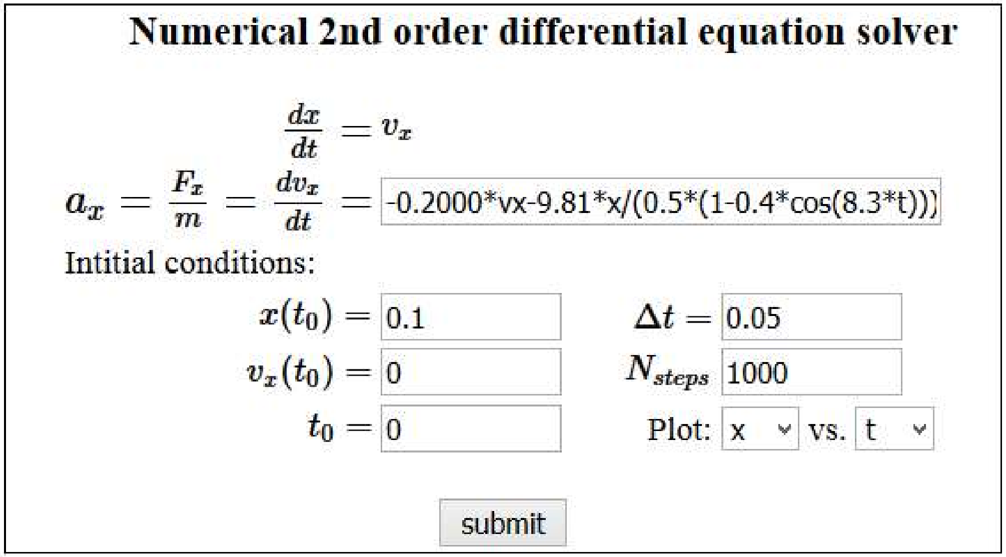
The solutions to a linear differential equation with periodic coefficients,

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+c(x)y=d,$$

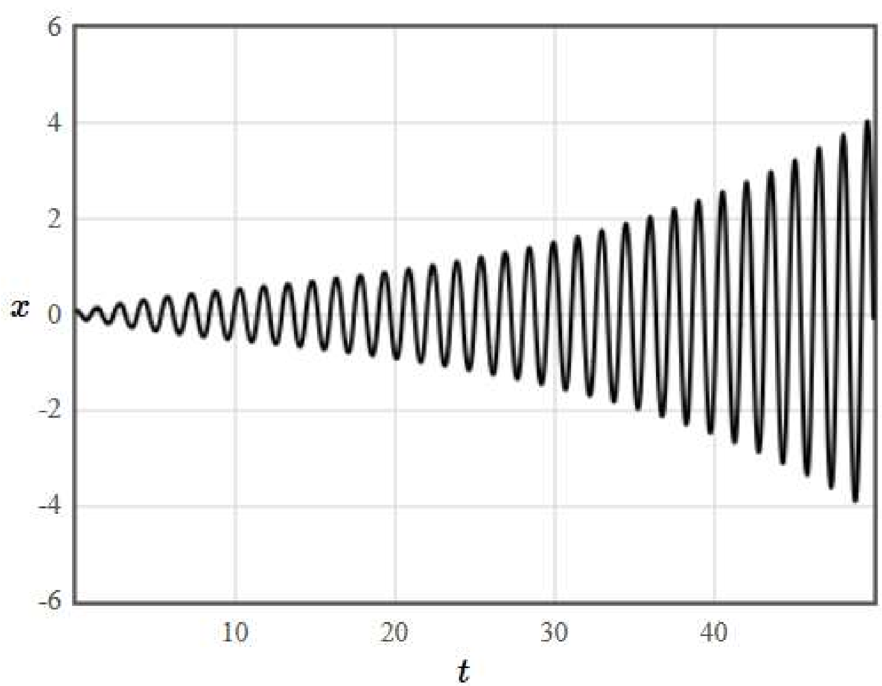
have the form,

$$e^{ikx}u_k(x) \qquad \text{where} \qquad u_k(x)=u_k(x+a)$$

Swing



$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{mg}{l(1 - A\cos(\omega t))}x = 0.$$



For some parameters there are periodic solutions (band).

For some parameters there are exponentially growing and decaying solutions (bandgap).

Translational symmetry

The normal modes are eigenfunctions of the translation operator

The normal modes have Bloch form.

$$\xi(x) = e^{ikx} u_k(x) \qquad \text{where} \qquad u_k(x) = u_k(x+a)$$

$$Te^{ikx} u_k(x) = e^{ik(x+a)} u_k(x+a) = e^{ika} e^{ikx} u_k(x)$$