NOISE DRIVEN FLUCTUATIONS OF JOSEPHSON JUNCTION SERIES ARRAYS

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Abstract

Series arrays of Josephson junctions have potential utility as mm-wave/sub mm-wave oscillators. Here the shunted junction model is used to analyze the noise driven fluctuations of such arrays. These fluctuations are responsible for the linewidth of the oscillations that the junctions produce. We show that there are two types of fluctuations, each of which make their own characteristic contribution to the power spectrum. The form of these fluctuations is calculated in the limit of small noise and we show that the fluctuations increase as a dynamical instability is approached.

Introduction

When a voltage is present across a Josephson junction, high frequency oscillations of the supercurrent arise. This is the ac Josephson effect, and it was long ago suggested that this effect could be exploited to construct a rapidly tunable rf generator.[1] The simplest generator of this type consists of a single Josephson junction. Unfortunately single junction generators have the practical limitations that their impedance and output power are impractically small. For this reason there has been an interest in fabricating rf generators from series arrays of Josephson junctions. If all of the junctions of an N junction array oscillate coherently, then the array will have N times the impedance and N^2 times the output power as a single junction.[2] Ideally these arrays generate a single, stable frequency, but inevitable fluctuations give the output a finite linewidth. It is the form of the fluctuations and the resulting linewidth that are the focus of this paper. Assuming that the fluctuations at each junction are independent, then the voltage fluctuations across the whole array will scale like $\sqrt{N}$.[3] This narrowing of the linewidth in coherent arrays has been observed experimentally.[4] Previous theoretical discussions of the linewidth focused on voltage fluctuations which have the form $V(t)=V_o(t+\psi(t))$, where $V_o(t)$ is the noise-free voltage across the junctions and $\psi(t)$ is a fluctuating phase. In phase space $\psi(t)$ corresponds to fluctuations along the noise-free trajectory. Here we present an analysis which shows that one additionally has further fluctuations; in this case the voltage has the form $V(t)=V_o(t+\psi(t))+V_T(t)$, where $V_T(t)$ describes fluctuations transverse to the noise-free trajectory. Although the transverse fluctuations can often be neglected, in the vicinity of certain dynamical instabilities they can make a significant contribution to the voltage fluctuations and thus to the power spectrum of the oscillating array.

Basic Circuit and Model

The general circuit diagram of the array we are considering is shown in Fig. 1. It is a series array of current biased Josephson junctions shunted by a matched resistive load. Typically the load would be a mixer or a transmission line. Both the generator and the load must be included in the analysis since it is the load that couples the junctions together and serves to phaselock them. The behavior of Josephson junctions is commonly modeled using the Shunted Junction Model.[5,6] Within this model the equations that describe the behavior of the circuit in Fig. 1 are,

$$\beta_c\ddot{\varphi}_k+\dot{\varphi}_k+\sin(\varphi_k)+\frac{1}{N}\sum_j\dot{\varphi}_j = \xi_k(t)+\xi_L(t) \qquad k=1,\ldots,N \tag{1}$$

We have used the usual reduced units, measuring current in units of the critical current, I_c, voltage in units of I_cR_N, and time in units of $\hbar/2eI_cR_N$. The resistance of the load was taken to be equal to the resistance of the array, $R_L=NR_N$. Here N is the number of junctions, R_N is the appropriate shunt resistance of the junctions, $\beta_c=(2eI_cR_N{}^2C_j)/\hbar$ is a dimensionless measure of the capacitance, C_j, of the junctions, I_L is the load current, and I_B is the applied bias current. The φ_k's are the differences in the phases of the quasiclassical, superconducting wavefunctions on the two sides of the junctions, and $\xi_k(t)$ and $\xi_L(t)$ represent the random noise generated in the kth junction and the load respectively. One unavoidable source of noise is the Johnson noise associated with the resistors in the equivalent circuit of the array. In this case the noise sources act independently at each resistor, $\langle\xi_k(t)\rangle=0$, $\langle\xi_L(t)\rangle=0$, $\langle\xi_k(t)\xi_{k'}(t')\rangle=T_B\delta_{kk'}\delta(t-t')$, $\langle\xi_L(t)\xi_L(t')\rangle=T_B\delta(t-t')/N$, and $\langle\xi_k(t)\xi_L(t')\rangle=0$, where T_B is the normalized bath temperature, $T_B=4ek_BT/\hbar I_c$.

As we mentioned above this circuit has potential as an rf generator if all of the junctions oscillate in the in-phase solution, $\varphi_k=\varphi_o$. In the absence of noise the equation for the in-phase solution is equivalent to the following single junction equation,

$$\beta_c\ddot{\phi}_o + 2\dot{\phi}_o + \sin(\varphi_o) = I_B \tag{2}$$

Recently we analyzed the in-phase solution and showed that for $\beta_c>0$, arrays with pure resistive loads can be stable.[7] Previous work at $\beta_c=0$ had shown that an inductive load was needed to stabilize the coherent oscillations.[1] Unfortunately the in-phase state is not always stable. Other stable solutions are known to exist for these equations. The form of these other solutions is discussed in Ref. [7]. In this paper we will assume that the array is operating in a regime where the in-phase solution is stable, and we discuss the fluctuations about this stable solution.

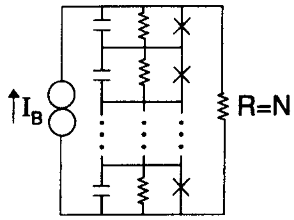

Figure 1. Circuit diagram for a Josephson junction rf generator.

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Equation (1) is invariant with respect to translations in time and is thus called an autonomous system. All autonomous systems subjected to noise exhibit two fundamentally different types of fluctuations. We call the first type phase fluctuations. These occur as random noise kicks the system along its noise-free solution. For instance the phase fluctuations modify the voltage across the array so that it takes the form, $V(t)=V_o(t+\psi(t))$, where $V_o(t)$ is the noise-free voltage and $\psi(t)$ are the phase fluctuations. The effect that the phase fluctuations have on the linewidth can be seen by writing the noise-free voltage in a Fourier series, $V_o(t)=\sum_{\omega} a_{\omega} e^{i\omega t}$. With the phase fluctuations included $V(t)$ becomes, $V(t)=\sum_{\omega} a_{\omega} e^{i\omega(t+\psi)}$. Here each Fourier component of the noise-free solution is multiplied by a fluctuating factor, $e^{i\omega\psi}$. Thus the lineshape of the Fourier component at frequency ω is determined by the square of the Fourier transform of $e^{i\omega\psi}$. The second type of fluctuations come from the noise kicking the system transverse to the noise-free solution. Mathematically the transverse fluctuations are described by adding a term to the noise-free voltage, $V(t)=V_o(t)+V_T(t)$. As we show below, near certain instabilities (called local bifurcations) the transverse fluctuations become very large. When this happens the transverse fluctuations can make a significant contribution to the power spectrum of the oscillations.

Both types of fluctuations can be visualized in the phase space for the dynamics. A representation of the 2N dimensional phase space is illustrated in Fig. 2. Periodic solutions (such as the in-phase solution) describe a closed trajectory in phase space. Phase fluctuations, ψ, describe deviations tangent to this trajectory while transverse fluctuations, x, describe deviations in the 2N-1 directions transverse to the trajectory.

The simplest approximation that has been used to analyze single junctions is to ignore the transverse fluctuations and to assume that,

$$\psi=\int \Xi(t)dt, \tag{3}$$

where $\Xi(t)$ is Gaussian white noise, $\langle\Xi(t)\rangle=0$, $\langle\Xi(t)\Xi(t+\tau)\rangle=\kappa\delta(\tau)$. This leads to the familiar result that the oscillations have a Lorentzian lineshape.[5,8] In the next section we find a better approximation for the fluctuations in the limit of small noise.

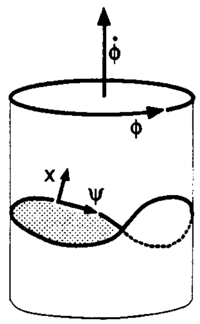

Figure 2. Representation of the in-phase solution in phase space indicating how the phase fluctuations and the transverse fluctuations describe the deviations from the in-phase solution. For this system the dynamics take place on an N dimensional cylinder where the variables φ_k are periodic and the variables $\dot\varphi_k$ can take on any value. The in-phase solution describes a closed trajectory on this N-cylinder. The phase fluctuations are the deviations tangent to the in-phase trajectory and the transverse fluctuations are the deviations transverse to the trajectory.

Small Noise Approximation

When the noise is small, the deviations from the in-phase solution are likewise small so we consider a solution to Eqn. (1) of the form $\varphi_k=\varphi_o(t)+\eta_k$. The linearized equations for η_k are,

$$\beta\ddot\eta_k + \dot\eta_k + \cos(\varphi_o)\eta_k + \frac{1}{N}\sum_j \dot\eta_j = \xi_k(t) + \xi_L(t) \tag{4}$$

This approximation relies on linearizing the equations about a stable, noise-free solution. This implies that the results obtained here are only valid when the noise is sufficiently small that it does not perturb the system too far from the noise-free solution. For example, it has been reported that larger noise can lead to hopping between coexisting stable solutions.[9] The results presented here would not be valid in this case.

We can greatly simplify Eqn. (4) by taking advantage of the fact that any permutation, $\varphi_j\leftrightarrow\varphi_k$, leaves Eqn. (1) unchanged. We transform to the natural coordinates of this system, which are the mean coordinate $\vartheta=(1/N)\Sigma\eta_k$, and the N-1 relative coordinates, $\zeta_k=\eta_k-\eta_{k+1}$. The linearized equations then become,

$$\beta_c\ddot\zeta_k + \dot\zeta_k + \cos(\varphi_o)\zeta_k = \xi_k-\xi_{k+1} \tag{5a}$$

$$\beta_c\ddot\vartheta + 2\dot\vartheta + \cos(\varphi_o)\vartheta = \frac{1}{N}\sum_k \xi_k + \xi_L \tag{5b}$$

This transformation decouples all N coordinates in the problem. Special attention should be given to ϑ since it is the variable that describes the fluctuations across the entire array. These are the fluctuations that appear in the generator output.

Neglecting noise for a moment, consider the homogeneous solutions to Eqn. (5). These solutions describe how an impulse perturbation to the in-phase solution evolves. Since Eqn. (5) is linear with periodic coefficients we know from Floquet theory that the homogeneous solutions have the form, $e^{\rho t}\chi(t)$, where $\chi(t)$ is a periodic function with the same period as $\varphi_o(t)$. The quantity, ρ, is called Floquet exponent and $\mathrm{Re}(\rho)$ specifies the rate of decay of an impulse perturbation. In particular the two homogeneous solutions to Eqn. (5b) are,

$$\vartheta_{\parallel}=\dot\phi_o \qquad \text{and} \qquad \vartheta_{\perp}=\dot\phi_o(t)\int^t \frac{e^{-2t'/\beta_c}}{\dot\phi_o^2(t')}dt' \tag{6}$$

Here $\vartheta_{\parallel}$ is periodic ($\rho=0$) and corresponds to a perturbation along the in-phase solution. When a small perturbation of the form $\varepsilon\vartheta_{\parallel}$ is added to the in-phase solution, the result is equivalent to translating the origin of time by the small quantity ε, $\varphi_o+\varepsilon\dot\varphi_o \approx \varphi_o(t+\varepsilon)$. Because of this property $\vartheta_{\parallel}$ plays a special role in the analysis, namely, we identify the inhomogeneous solution of Eqn. (5b) associated with $\vartheta_{\parallel}$ as the phase fluctuations. When the in-phase solution is stable all of the other homogeneous solutions to Eqn. (5) decay exponentially ($\rho<0$). These other inhomogeneous solutions of Eqn. (5) are then identified as the transverse fluctuations.

Solving for the inhomogeneous solutions to Eqn. (5) in terms of the homogeneous solutions, and inverting the transformation to the relative and mean coordinates, one obtains[10] an approximate solution to Eqn. (1) of the form $\varphi_k=\varphi_o(t+\psi)+x_k$ where

$$\psi = -\int^{t} \frac{\left(\frac{1}{N}\sum_k \xi_k+\xi_L\right)\dot{\varphi}_o(t')}{\beta_c e^{-2t'/\beta_c}} \int^{t'} \frac{e^{-2t''/\beta_c}}{\dot{\varphi}_o^2(t'')} dt''dt' \qquad (7a)$$

$$x_k=\vartheta_\perp \int^{t} \frac{\left(\frac{1}{N}\sum_k \xi_k+\xi_L\right)\dot{\varphi}_o}{\beta_c e^{-2t'/\beta_c}} dt' - \frac{1}{N}\sum_{q=1}^{k-1} q\zeta_q + \frac{1}{N}\sum_{q=k}^{N-1}(N-q)\zeta_q \qquad (7b)$$

Here ζ_q are the inhomogeneous solutions to Eqn. (5a) which, unfortunately, must be calculated numerically. The procedure for doing this was described in our earlier paper.[7] For large bias currents, $\dot{\varphi}_o$ is essentially a constant and can be taken out of the integral in the expression for the phase fluctuations. The resulting expression provides a correction to the simpler approximation for the phase fluctuations given above in Eqn. (3). In this limit the amplitude of the phase fluctuations is inversely proportional to the bias current. For lower bias currents where $\dot{\varphi}_o$ cannot be taken to be a constant, the phase fluctuations must be determined numerically from Eqn. (7).

One experimentally accessible variable that exhibits both phase and transverse fluctuations is the voltage across the kth junction of the array. In the normalized units this voltage is, $V_k(t)=\dot{\varphi}_o(t+\psi)+x_k$. Phase fluctuations enter the voltage oscillations through the term, $\dot{\varphi}_o(t+\psi)$, and cause a broadening of the linewidth of these oscillations. As we stated above the lineshape of the Fourier component of $\dot{\varphi}_o$ at frequency ω is given by the square of the Fourier transform of the fluctuating function, $e^{i\omega\psi}$, where ψ is determined from Eqn. (7). All of the transverse fluctuations of this voltage have the same form. They are inhomogeneous solutions to noise driven equations whose homogeneous solutions are of Floquet form. Solutions of this sort have been studied before.[11] It was shown that these terms have a power spectrum that is the sum of Lorentzians whose shape is determined by the quantity ρT, where ρ is the Floquet exponent of a homogeneous solution, and T is the period of the noise-free oscillations. There is one Lorentzian contributed at each Fourier component of $\dot{\varphi}_o$ by every homogeneous solution. The Lorentzian's width is proportional to ρT and its amplitude is inversely proportional to ρT.

If any $Re(\rho_k)$ becomes greater than zero, then a small perturbation will grow exponentially. Such an event is called a local bifurcation and signals an abrupt change in the dynamics. As one of these bifurcations is approached ($Re(\rho_k)\to 0$), one or more of the Lorentzians becomes large and narrow and makes an important contribution to the power spectrum. This phenomena, which has been observed experimentally in other systems (including chemical, electrical and optical experiments),[12] is called the noisy precursor to a bifurcation. It has been demonstrated that the power spectrum has certain universal scaling properties close to the bifurcation.[11] The bifurcations of the in-phase solution correspond to instabilities where all of the junctions no longer oscillate identically. At this instability (N-1)! new, symmetry related solutions appear.[7] Thus the power spectrum of an individual junction of the array consists of a peaks that correspond to the basic oscillations of the junctions which have been broadened by phase fluctuations plus noise bumps which correspond to the decay of the transverse perturbations. When an instability is approached the noise bumps become more prominent in the power spectrum.

For the rf generator application it is most important to consider how the fluctuations effect the total voltage across the array. For small noise the voltage across the whole array is,

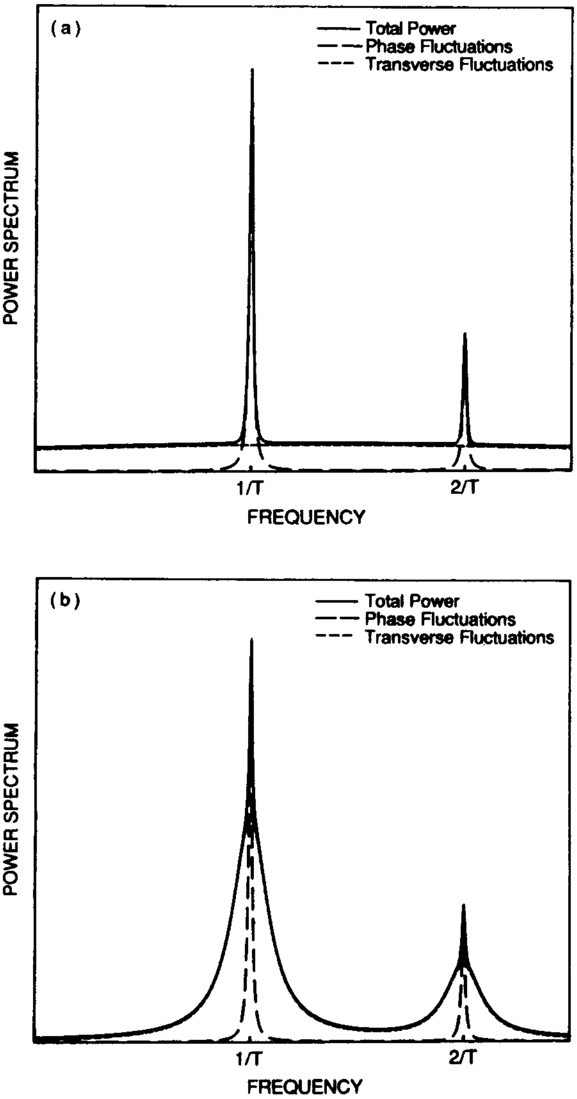

Figure 3. Power spectrum of the total voltage across the array showing the contributions of the phase and transverse fluctuations. 3a - Large T/β_c where the transverse fluctuations make an nearly flat contribution to the power spectrum. 3b - Small T/β_c where the contributions to the power spectrum are peaked around the fundamental and the harmonics of the basic Josephson oscillations.

$$V(t) = N\phi_o(t+\psi) + \dot{\vartheta}_\perp(t)\int^t \frac{(\frac{1}{N}\sum_k \xi_k+\xi_L)\phi_o(t')}{\beta_c e^{-2t'/\beta_c}}dt' \qquad (8)$$

Note that the transverse fluctuations of the total voltage depend only on the mean coordinate, ϑ. This leads to the surprising result that the noise bumps which correspond to instabilities where the array loses coherence do not appear in the power spectrum of the total voltage across the array. The power spectrum of the total voltage consists of two components. The first is the broadening of the linewidth of the oscillations due to phase fluctuations which was described above. The other component is the noise bump due to the transverse fluctuations. This component of the spectrum contributes Lorentzians centered at the Fourier components of $\dot{\phi}_o$. The amplitude and width of these Lorentzians are governed by the dimensionless quantity, ρT, where $\rho=-2/\beta_c$. For small β_c and large T (large oscillation periods correspond to low bias currents), the noise bumps will be very broad so the transverse fluctuations contribute an essentially flat component to the power spectrum. In this case the linewidth will be due entirely to the phase fluctuations. This is illustrated in Fig. 3a. For large β_c or large bias currents (small T), the transverse fluctuations contribute narrow Lorentzians at the Fourier components of $\dot{\phi}_o(t)$ which add to the phase fluctuations to make up the linewidth (see Fig. 3b). In our previous work we showed that the in-phase state of a resistively shunted array is maximally stable for $\beta_c=0.75$ and $I_B=2.3$. This maximally stable state produces a power spectrum qualitatively like that shown in Fig. 3a.

Conclusions

The total voltage across an array of coherently oscillating Josephson junctions exhibits two fundamentally different types of fluctuations, each of which makes its own characteristic type of contribution to the power spectrum. Phase fluctuations broaden the peaks in the power spectrum that correspond to the basic oscillations of the junctions and are primarily responsible for the linewidth of these oscillations. Transverse fluctuations contribute Lorentzian shaped noise bumps to the power spectrum at the fundamental and harmonics of the basic Josephson oscillations. These noise bumps become larger and narrower as T/β_c increases, making a contribution to the linewidth for large β_c and large bias currents. The approach of the instability which corresponds to the array losing coherence is responsible for the appearance of noise bumps in the power spectra of the individual junction voltages but the impending instability does not effect the power spectrum of the total voltage across the array. Finally we emphasize that all of this analysis has assumed that the in-phase solution was stable and that the noise was small. When either of these conditions are not met nonlinear effects will will have to be taken into account.

Acknowledgements

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References

[1] See, A. K. Jain, K. K. Likharev, J. E. Lukens, and J. E. Sauvageau, Phys. Rep. , vol. 109, p. 310, 1984 and references therein.

[2] D. R. Tilley, Phys. Lett. , vol. 33A, p. 205, 1970.

[3] T. D. Clark, Phys. Lett., vol. 27A, p. 585, 1968.

[4] C. Varmazis, R. D. Sandell, A. K. Jain, and J. E. Lukens, Appl. Phys. Lett., vol. 33, p. 357, 1978.

[5] K. K. Likharev, Dynamics of Josephson Junctions and Circuits. New York: Gordon and Breach Science Publishers, , 1986.

[6] T. Van Duzer and C. W. Turner, Principles of Superconductive Devices and Circuits, New York: Elsevier, 1981, p.170.

[7] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Appl. Phys. Lett., vol. 52, p. 1619, 1988; P. Hadley, M. R. Beasley, and K. Wiesenfeld, to be published in Phys. Rev. B.

[8] J. Stewart, Proc. IRE, p. 1539, 1954.

[9] R. F. Miracky, J. Clarke, and R. H. Koch, Phys. Rev. Lett., vol. 50, p. 856, 1983.

[10] P. Hadley, Stanford University Ph.D. thesis.

[11] K. Wiesenfeld, J. Stat. Phys., vol. 38, p. 1971, 1985.

[12] See for example, K. Wiesenfeld, "Period Doubling Bifurcations, What Good Are They?" in Noise in Nonlinear Systems, Eds: F. Moss and T. V. E. McClintock, Cambridge: Cambridge University Press, 1988.