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Stability of Coherent Oscillations in Josephson Junction Arrays

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When series arrays of Josephson junctions are included in circuits there exists the possibility that the junctions will phaselock and oscillate coherently. We present a method to study the stability of the phase coherent state based on a determination of the eigenvalues of the return map associated with the dynamics. The eigenvalues indicate how close the coherent state is to an instability. Therefore knowledge of the eigenvalues allows us to find the operating point which maximizes the stability of the coherent state. The power of our method is that arbitrarily large arrays are handled as easily as small arrays. For series arrays of N junctions with small shunt capacitances we show that the coherent state always loses stability in the same way, via a codimension N-1 bifurcation. We apply this method to Josephson junction arrays proposed as rf generators.

When the current passing through a Josephson junction exceeds the junction's critical current, the supercurrent flowing through the junction oscillates at a frequency that is proportional to the voltage across the junction. This is the familiar ac Josephson effect. When the oscillating supercurrents of a number of junctions interact with each other there exists the possibility that the junctions will phaselock and oscillate coherently. In this work we study the mutual phaselocking of dc biased series arrays of Josephson junctions shunted by a load impedance. In these circuits the voltage across the junctions generates a high frequency load current. This current flows through all of the junctions and serves to couple them. In the Resistively Shunted Junction (RSJ) model the equations that describe the behavior of the circuit are [1]

$$\begin{aligned} &\beta_c\ddot{\varphi}_k + \dot{\varphi}_k + \sin(\varphi_k) + I_L = I_B && k=1,\dots,N \\ &\sum_{k=1}^{N} \dot{\varphi}_k = V && \\ &\dot{I}_L = f(V, I_L, y_1, \dots, y_M) && (1) \\ &\dot{y}_j = g_j(V, I_L, y_1, \dots, y_M) && j=1,\dots,M \end{aligned}$$

Where the φ_k are the phase differences across the junctions, I_B is the bias current, I_L is the load current, V is the voltage across the array, the y_j's are the M variables that specify the state of the load, and f and g_j are functions that describe the dynamics of the load. The dimensionless parameters, $\beta_c=(2eI_cR_N^2C)/\hbar$ and $\beta_L=(2eI_cL)/\hbar$, characterize the capacitance and the inductance of the circuit.

In the coherent state all of the junctions oscillate with the same frequency and phase. In this state the array acts like a single junction. This makes it relatively simple to calculate the coherent solution. However, calculating the coherent solution is not enough; we also need to determine under what conditions that solution is stable. To study the stability of the coherent state we considered small perturbations to the coherent solution, $\varphi_k=\varphi^c+\varphi_k{}^p$. Here the superscript c denotes the coherent state and the superscript p denotes the perturbation. Linearizing around the coherent state results in a set of 2N+M linear equations with periodic coefficients. Using these equations one can calculate how the perturbations evolve in time. If the perturbations diminish with time then the coherent state is linearly stable. If the pertrubations grow, the coherent state is linearly unstable.

A symmetry in the problem, namely the interchangability of the junctions, allows us to make a transformation that simplifies the linearized equations enormously.[2] We transform to the mean coordinate, $\vartheta=\Sigma\varphi_k{}^p$, and the relative coordinates, $\zeta_k=\varphi_k{}^p-\varphi_{k+1}{}^p$, (k=1,...,N-1). This transformation decouples the relative coordinates from all of the other variables in the problem. Further simplification results because all of the relative coordinates obey the same equation. Thus, because of symmetry, the analysis of the stability of the original 2N+M dimensional problem reduces to analyzing a set of 2+M coupled linear differential equations.

The remaining equations are solved by constructing the return map for this system. We employ the return map formalism because it provides a quantitative measure of the stability and allows us to find the operating point that maximizes the stability of the coherent state. A return map, in this context, can be understood as follows.[3] Any Nth order set of linear differential equations with periodic coefficients can be written in vector form as

$$\dot{\mathbf{u}} = \mathbf{F}(t)\,\mathbf{u} \qquad (2)$$

where **u** is an N dimensional vector and **F** is a time dependent NxN matrix which is periodic with period T. Since **F** is periodic, if **u**(t) is a solution of eq. (2) then $\mathbf{u}(t+T)$ must also be a solution. We define a time independent matrix, $\mathbf{A}$, by

$$\mathbf{u}(t+T) = \mathbf{A}\mathbf{u}(t) \qquad (3)$$

where $\mathbf{A}$ is called the return map. Repeated applications of the return map produce the solution at later times.

$$\mathbf{u}(t+nT) = \mathbf{A}^n\mathbf{u}(t) \qquad (4)$$

The magnitude of the eigenvalues of the return map provide a measure of the stability of the coherent state. If all of the eigenvalues of $\mathbf{A}$ lie inside the unit circle then the perturbations diminish with time and the coherent state is stable. If one or more eigenvalue crosses the unit circle the perturbations will grow with time and the coherent state is unstable. When the eigenvalue crosses the unit circle we say that the system experiences a bifurcation.

To demonstrate these techniques we calculated the stability of the coherent state for the circuit shown in Fig. 1. The circuit consists of a series array of N Josephson junctions shunted by a series LC load where $\beta_c=1/(2N)$ and $\beta_l=N/2$. For simplicity we have taken the junction capacitance to be equal to zero. Figure 1 displays the magnitude of the eigenvalues of the return map as a function of bias current. Note that this figure is independent of N. One of the eigenvalues is always equal to one. This is always true for autonomous systems such as the one we are considering. The coherent state is stable for bias currents where all of the magnitudes of the eigenvalues are less than or equal to one. In this case the coherent state loses stability at a bias current of 2. At this point the circuit experiences a

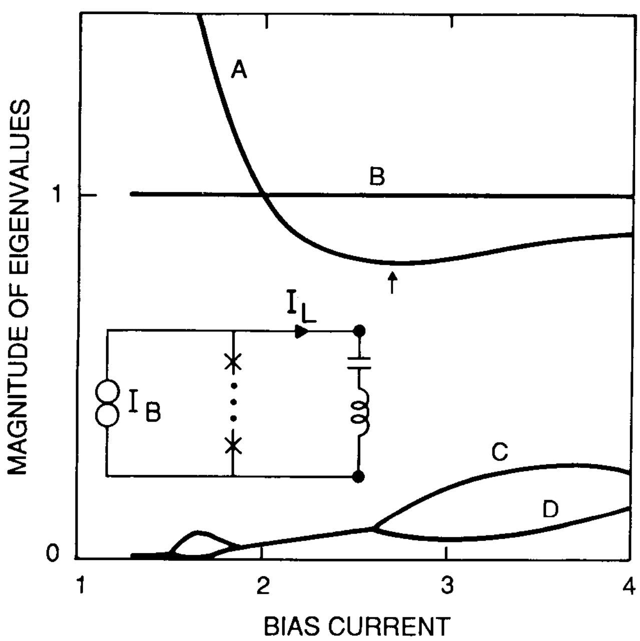

Fig. 1. The magnitude of the eigenvalues for the circuit in the inset is plotted vs. bias current. A is the (N-1)-fold degenerate eigenvalue that crosses +1 at the saddle-node bifurcation. B is a single eigenvalue equal to +1 characteristic of autonomous systems. C and D are eigenvalues associated with the degrees of freedom of the load.

saddle-node bifurcation where N-1 of the eigenvalues cross the unit circle at +1 simultaneously. This is the only allowed bifurcation that the coherent state can suffer when the junctions have zero capacitance because the expression for the degenerate eigenvalue is

$$\lambda = \exp\left(-\int_0^T \cos(\varphi^c(t))\, dt\right) \qquad (5)$$

which is obtained by integrating the linearized equations, is always a real, positive quantitiy. The arrow in the figure indicates the operating point where the coherent state is most stable. For lower bias currents the circuit approaches the bifurcation and for higher bias currents the load current oscillations are smaller and the phases of the junctions don't couple as strongly.

When the coherent state loses stability (I_B <2) the array enters an antiphase state where each junction oscillates at the same frequency but the sum of the phases adds to zero. [4] Our simulations show that the coherent solution is stable when the impedance of the load is inductive at the fundamental Josephson frequency and that the anti-phase solution is stable when it is capacitive. This confirms the perturbation calculations for high bias currents of Jain et al. [5]

In summary we have described a method that provides a quantitative measure of the stability of the coherent state of series Josephson junction arrays. By taking advantage of a symmetry we were able to reduce the order of the equations that needed to be solved by N-1, where N is the number of junctions in the array. The reduced equations were then analyzed using the return map formalism. This allowed us to determine the operating point that maximized the stability of the coherent state. Finally, for arrays of small capacitance junctions we showed that the coherent state always loses stability via a saddle-node bifurcation where N-1 eigenvalues cross the unit circle simultaneouly.

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REFERENCES

1) It is customary to measure time in units of $\hbar/(2eI_cR_N)$, current in units of I_c, and voltage in units of I_cR_N, where I_c is the critical current of the junction and R_N is the shunt resistance.

2) K. Wiesenfeld, Private Communication.

3) See, for example, Nayfeh, A.H., and D.T. Mook, *Nonlinear Oscillations* (Wiley, New York, 1979).

4) P. Hadley and M.R. Beasley, *Appl. Phys. Lett.* **50** (1987) p. 621.

5) A. K. Jain, K. K. Likharev, J. E. Lukens, and J. E. Sauvageau, *Phys. Rep.* **109** (1984) p. 310.