Attractor Crowding in Oscillator Arrays

Kurt Wiesenfeld and Peter Hadley[a]

Physics Department, Georgia Institute of Technology, Atlanta, Georgia 30332

(Received 23 December 1988)

We describe a novel feature of certain arrays of N coupled nonlinear oscillators. Specifically, the number of stable limit cycles scales as $(N-1)!$. To accommodate this huge multiplicity of attractors, the basins of attraction crowd even more tightly in phase space with increasing N. Our simulations show that for large enough N, even minute levels of noise cause the system to hop freely among the many coexisting stable attractors.

PACS numbers: 05.45.+b, 05.40.+j

Recent advances in understanding the dynamics of nonlinear systems possessing a few degrees of freedom have encouraged researchers to try to extend these results to systems with many degrees of freedom. In the main, these studies are motivated by the desire to understand the dynamical behavior of systems which have nontrivial spatial structure—that is, systems traditionally described by partial differential equations, though coupled ordinary differential equations, coupled maps, and cellular automata are also used. As models for spatially extended systems, the main interest is for *very* large N. In fact, progress has been slow, except in those cases where the dynamics reduces to a low-dimensional phase space. Part of the difficulty stems from the fact that systems often exhibit dynamical behavior which is *intrinsically* high dimensional. Phenomenological studies of such systems have revealed a number of intriguing high-dimensional behaviors, including robust space-time intermittency,[1-3] spatial periodic doubling,[3] phase organization and dynamical selection of minimally stable states,[4,5] self-organized criticality,[6-12] and phase-locking plateaus.[13] Also, in certain coupled map lattices, a global picture relating regimes of qualitatively different behavior is emerging.[14]

In this paper, we describe a new high-dimensional behavior observed in arrays of N coupled oscillators. Typically, the array settles down into a stable periodic solution; however, this solution can coexist with other stable periodic solutions. Each stable solution is an attractor, so that nearby trajectories in phase space evolve toward the attractor. For the oscillator arrays that we consider here, the number of coexisting attractors increases explosively with increasing N. As a result of this explosive growth in the number of stable solutions, the attractors crowd ever more tightly in phase space. For large enough N, the crowding is so severe that even a very small level of external noise causes the system to hop among the stable solutions. Below we present simulations that show the onset of this hopping as N is increased.

We base our description of attractor crowding on a combination of numerical simulations of two nonlinear oscillator systems—one of ordinary differential equations, the other iterative maps—and rigorous constraints imposed by the underlying symmetry of the governing dynamical equations. Though the symmetry considerations give us a rigorous framework in which to understand the observed dynamics, we will argue that the existence of a precise symmetry *is not* fundamental to the phenomenon of attractor crowding.

We first observed the phenomenon of attractor crowding while studying coupled arrays of Josephson junctions. These arrays have potential in a variety of applications,[15] e.g., as generators of millimeter wave radiation, as sensitive parametric amplifiers, and as voltage standards. For example, the National Institute of Standards and Technology has tested circuits involving 2076 Josephson junctions.[16]

As part of a systematic study of the dynamics of series arrays of Josephson junctions, we consider the circuit shown in Fig. 1. The governing equations are, in dimensionless form,[17,18]

$$\beta\ddot{\phi}_k+\dot{\phi}_k+\sin\phi_k+I=I_B,\quad k=1,2,\ldots,N, \tag{1a}$$

$$L_N\dot{I}+\int I/C_N\,dt=\sum_k\dot{\phi}_k. \tag{1b}$$

Here, ϕ_k represents the phase difference of the macroscopic wave function across the kth junction, I is the current flowing through the inductor-capacitor load, and I_B is a constant bias current. The dimensionless parameters L_N, C_N, and β are measures of the inductance of the load, capacitance of the load, and capacitance of the junctions, respectively. Equation (1a) represents current conservation, while Eq. (1b) equates the total voltage across the array to the voltage across the load. (In these units the voltage across the kth junction is just $\dot{\phi}_k$.) Each junction is coupled to the rest of the array by the current I; thus, the magnitude of I is a measure of the coupling strength. In order to keep this coupling constant while the number of junctions is varied, the normalized inductance and capacitance are chosen to be $L_N=L/N$ and $C_N=NC$, where L and C are constants. Notice that since the series array shares a common load,

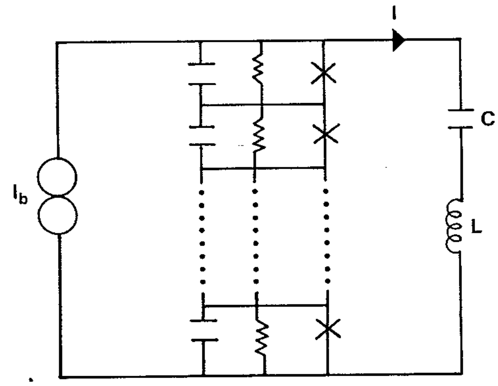

FIG. 1. Circuit schematic for series array of Josephson junctions described by Eqs. (1).

each junction is coupled to all the others; this form of coupling is also familiar from mean-field descriptions of spatially extended systems. This high connectivity is significant in what follows.

This dynamical system has been studied in some detail.[15,17-19] For practical applications it is desirable that the junctions oscillate identically, $\phi_k(t)=\phi_0(t)$ for all k. The range of parameters over which such "in-phase" solutions are stable has been determined numerically, both for this[17,18] and for a variety of other loads.[18,19] In addition to the in-phase solution, this system can have other stable solutions. For example, there are the *antiphase* solutions[17,18] which have the following properties: (i) Each $\dot{\phi}_k$ is periodic, with the same period T, and (ii) $\phi_k \neq \phi_j$, for j different from k.

From the viewpoint of dynamical systems, these antiphase solutions are of particular interest, because they occur with extremely high multiplicity. This follows rigorously from the symmetry of Eqs. (1): These are invariant with respect to any interchange $\phi_k \leftrightarrow \phi_j$. It follows that if X is a solution vector,

$$X=(\phi_1,\dot{\phi}_1,\phi_2,\dot{\phi}_2,\ldots,\phi_N,\dot{\phi}_N,I,\dot{I})\,,$$

then so is the vector with *any* permutation of the $\{\phi_k\}$. In the case of an in-phase solution, these $N!$ vectors all describe the same phase-space orbit, but for an antiphase state almost every permutation gives a distinct limit cycle. We say "almost" because, as it happens, the observed antiphase state can have some residual symmetry[17,18]; nevertheless, the condition $\phi_k \neq \phi_j$, for $j \neq k$ always obtains. This implies that there are at least $(N-1)!$ distinct limit cycles in phase space. This high multiplicity of attractors is the key to understanding attractor crowding.

Viewed as a function of N, this represents an explosive growth in the number of limit cycles. For the Josephson junction array, these attractors crowd ever more closely, so that the distance between them becomes vanishing small as $N\to\infty$. To see this, note that each variable ϕ_k is defined on the circle, so that the subspace $\{\phi_k\}$ has total volume $(2\pi)^N$; each basin of attraction thus has volume $(2\pi)^N/(N-1)!$. (This is an upper bound, since in general there are additional attractors in the phase space.) Thus, the characteristic linear dimension of each basin is $\sim V^{1/N}$. In the limit of large N, this distance between attractors vanishes as $1/N$.

Note that this argument depends on only gross features of the dynamical system: In particular, the form of the coupling and the underlying symmetry lead to large multiplicities. In fact, the presence of strict symmetry is not essential; rather, it allows an easy counting of attractors. We will return to this point later.

An important ramification of attractor crowding is that, for sufficiently large arrays, even extremely low levels of external noise can induce hopping among the coexisting attractors. In order to study this effect, we have run simulations on a second system that also displays attractor crowding, namely the set of coupled circle maps.

$$\phi^{(k)}\to\phi^{(k)}+\omega+A\sin(\phi^{(k)})+\frac{\sigma}{N}\sum_j\sin(\phi^{(j)})+\sqrt{\kappa}\,\xi^{(k)}\,,\qquad(2)$$

for $k=1,2,\ldots,N$. This system has the same permutation symmetry as Eqs. (1), and consequently the arithmetic of attractor counting is unchanged. We note that previous works[3,14] on coupled lattice maps with nearest-neighbor interactions has revealed cases of exponential growth in the number of attractors with array size N; in the systems studied here, the connectivity is greater —each oscillator is coupled to all the others— and this results in a factorial growth. In general, Eq. (2) displays both fully symmetric in-phase solutions, as well as various symmetry-broken ones. Specifically, for $\omega=2$, $A=0.5$, and $\sigma=1$, this system has stable, antiphase solutions in which each $\phi^{(k)}$ has the same winding number.

We have studied the behavior of Eq. (2) subject to random noise, with $\xi^{(k)}$ generated by choosing a uniformly distributed random number between -0.5 and $+0.5$ at each iterate, independently for each k. For each run, we pick random initial conditions, iterate without noise to make sure the system settles down to an antiphase attractor, then turn on the noise, iterating until the system crosses out of the original basin of attraction. (Since the ordering of the $\phi^{(k)}$ around the circle is preserved for each attractor, the signature of a basin boundary crossing was taken to be a change in this ordering.) For each value of N, fifty realizations were run in order to compute a mean escape time τ.

The results of our simulations are summarized in Fig. 2, where τ vs N is plotted for different values of noise intensity κ. The most striking feature is the rapid decrease of τ with N; in effect, there is a cutoff N^* above which the system escapes the attractor almost immediately. For larger κ this cutoff is sharp, while for smaller noise it is relatively gradual, making it difficult to assign a pre-

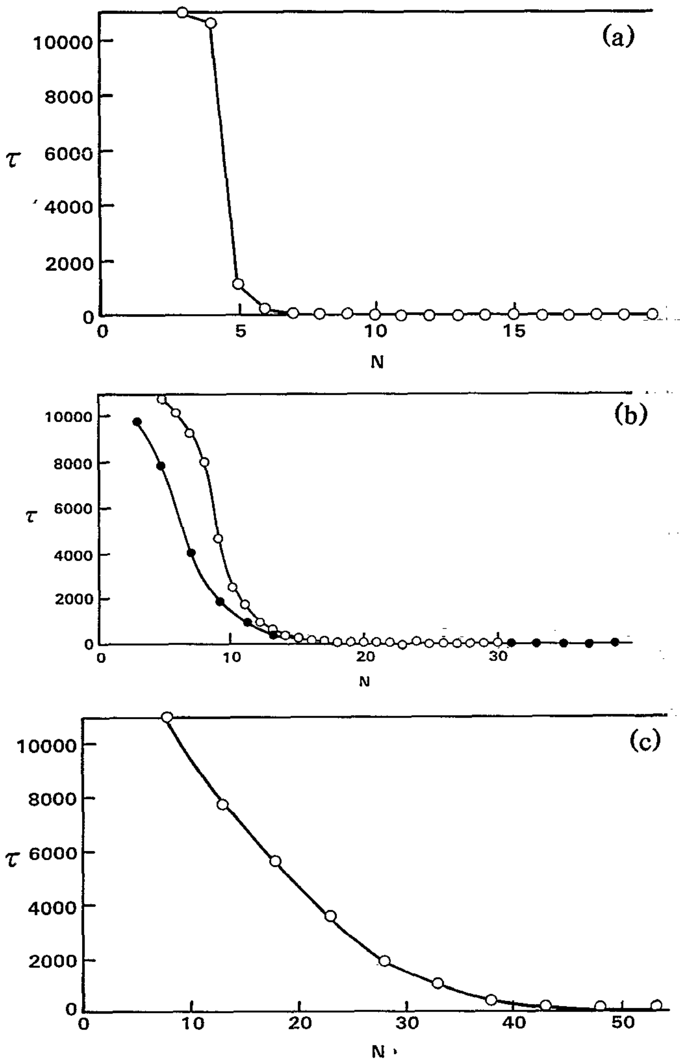

FIG. 2. Mean escape time τ vs number of maps N. (a) $\kappa=10^{-2}$, (b) $\kappa=10^{-4}$, and (c) $\kappa=10^{-6}$.

cise value for N^*. As expected, N^* increases with decreasing noise strength.

Qualitatively, when $N>N^*$, the system hops randomly from attractor to attractor. In this regime, the characteristic dimension of a basin is comparable to (or smaller than) the typical noise kick; consequently, we expect the details of the deterministic dynamics to be irrelevant insofar as this phase-space diffusion is concerned. To test this, we ran simulations in which the deterministic dynamics was switched off when the noise was turned on. That is, once the system settled down into an antiphase solution, the ensuing dynamics was a pure random walk:

$$\phi^{(k)} \rightarrow \phi^{(k)} + \sqrt{\kappa}\xi^{(k)}.$$

The result for the mean escape time (determined as before) versus N is shown by the filled circles in Fig. 2(b). Note that, except for very small N, the two curves are compatible. The fact that the filled circles fall below the open circles reflects the fact that escape is more difficult in Eq. (2), due to the presence of attracting dynamics near the noise-free orbits.

We return now to the role of symmetry. In the two examples cited above, the dynamics describe the interaction of N *identical* elements. This is reflected by a symmetry (or "equivariance") of the governing equations, which allow us to compute the number of antiphase solutions (and deduce their stability). However, this precise symmetry is not essential: Attractor crowding depends on the growing number of attractors, and this multiplicity does *not* require symmetry. In general, the effect of adding a sufficiently small symmetry-breaking term is crucial only near a bifurcation point. Although such terms fundamentally change the nature of the bifurcation itself, away from a small neighborhood of the bifurcation point the number of attractors is unchanged. In general, the robust character of such gross dynamical features rests on correspondingly general features of the governing evolution equations, namely smoothness of the vector field and the fact that the attractors are hyperbolic.

Although the existence of symmetry is not essential for attractor crowding, we stress that the symmetric case (coupled with perturbation theory) plays a crucial theoretical role, since only in this case is the direct computation of the number of attractors possible. To appreciate this fact, suppose one wishes to count the number of attractors for a large-N array having no apparent symmetry. Operationally, one is faced with a monumental problem: How can we verify the existence of some $N!$ attractors by purely numerical means?

Finally, one may return to the original motivation for studying Eq. (1), and ask is there are any practical ramifications of Josephson junction arrays. As mentioned, the chief practical interest is in the existence (and stability) of the in-phase solution. When this attractor coexists with antiphase solutions, it may happen that the competition for phase-space volume simultaneously crowds the in-phase basin, leading to enhanced sensitivity to low-level noise of the in-phase state. Such a coexistence occurs in both array systems described in this paper. (A characteristic signature of coexistence of attractors is hysteresis in the measured current-voltage curve.[17]) Figure 3 shows the result of simulations of Eq. (2) with $\omega=1.45$, $A=0.5$, and $\sigma=1$, where the system was put initially in an attracting symmetric state, in this case the stable fixed point $\sin\phi^{(k)}=-\omega/(A-\sigma)$. As expected, the mean escape time from the in-phase state decreases with increasing N. This is a consequence of the antiphase attractors crowding the in-phase attractor. Whether the same phenomenon is present in the differential equations (1) is currently under study.

In conclusion, we have described an intrinsically high-dimensional behavior of certain coupled nonlinear oscillatory systems. The number of attractors in these sys-

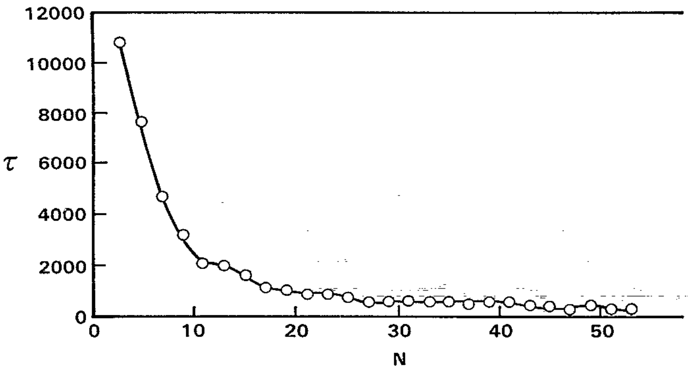

FIG. 3. Mean escape time τ from the symmetric (in-phase) attractor vs number of maps N; $\kappa = 6\times 10^{-2}$.

tems increases dramatically as the number of oscillators increases, resulting in the crowding of attractors in phase space. The crowding makes these systems sensitive to noise-driven hopping among the many coexisting attractors; large enough arrays will be sensitive to even extremely low levels of external noise.

We thank Mac Beasley, Paul Bryant, Ron Fox, John Guckenheimer, Bruce McNamara, and especially Jack Hale for useful discussions. P.H. thanks Georgia Tech for its hospitality this past summer. This work was supported by the Office of Naval Research under Contract No. N00014-88-K-0494.

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[a]Permanent address: Applied Physics Department, Stanford University, Stanford, CA 94305.

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