Phase locking of Josephson junction arrays

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We report the results of a stability analysis of coherent oscillations in series arrays of Josephson junctions with a matched resistive load. We find that arbitrarily large, dc biased arrays of Josephson junctions will phase lock most strongly when the capacitance parameter $\beta_c \approx 1$, and the bias current is about twice the critical current of the individual junctions.

Phase-locked series arrays of Josephson junctions are of interest as local oscillators in the microwave and millimeter wave regime.[1] These are essential components in any very high frequency electronic technology such as mm-wave receivers and spectrum analyzers. Arrays have the advantage over single-junction local oscillators in that the output power and source impedance can be increased to practically useful levels. The conditions under which arrays of Josephson junctions will lock coherently have been analyzed using perturbative techniques by Jain *et al.*[1] While extremely useful, their analysis is not applicable over all ranges of bias currents or for all the relevant ranges of junction capacitance. Here we employ a more general approach to calculate the stability of coherent oscillations in the practically important case of an arbitrarily large array of Josephson junctions shunted by a matched resistive load. A fuller development of this approach and its application to a wide variety of array circuits will be discussed elsewhere.[2]

To analyze this circuit (see the inset of Fig. 1) we model the junctions using the well-known shunted junction model.[3] In the usual reduced units[4] the equations for the array are

$$\beta_c \ddot{\varphi}_k + \dot{\varphi}_k + \sin(\varphi_k) + \frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_j = I_B$$

$$k = 1,2,\ldots,N, \tag{1}$$

where N is the number of identical junctions, I_B is the normalized bias current, β_c is a dimensionless measure of the capacitance of the junctions, and φ_k is the difference in the phases of the quasiclassical superconducting wave functions on the two sides of the k th junction. This model is a good approximation over a wide range of β_c, including weak-link (or superconducting/normal metal/superconducting-type) junctions ($\beta_c \ll 1$), shunted tunnel junctions ($\beta_c \approx 1$), and unshunted tunnel junctions at voltages below the energy gap ($\beta_c \gg 1$).

To analyze the coherent solution of such an array, we note that in this solution all of the junctions oscillate together, $\varphi_k = \varphi_0$. Therefore, the N equations for the array reduce to

$$\beta_c \ddot{\varphi}_0 + 2\dot{\varphi}_0 + \sin(\varphi_0) = I_B. \tag{2}$$

This is equivalent to the equation for a single junction with half the shunt resistance of the junctions in the array. Thus the calculation of the coherent state for an arbitrarily large array with a matched resistive load reduces to the equivalent calculation for a single junction.

To determine the stability of the coherent solution we consider small perturbations about it, $\varphi_k = \varphi_0 + \eta_k$. Linearizing around the coherent solution results in a set of linear differential equations with periodic coefficients:

$$\beta_c \ddot{\eta}_k + \dot{\eta}_k + \cos(\varphi_0)\eta_k + \frac{1}{N}\sum_{j=1}^{N}\dot{\eta}_j = 0,$$

$$k = 1,2,\ldots,N, \tag{3}$$

where $\varphi_0(t)$ is the periodic function, of period T, that solves Eq. (2).

We can greatly simplify the linearized equations by taking advantage of the permutation symmetry of the system. [Any permutation, $\eta_j \leftrightarrow \eta_k$, leaves Eq. (3) unchanged.] We transform to the natural coordinates of this system, which are the mean coordinate, $\vartheta = (1/N)\Sigma_{k=1}^{N}\eta_k$, and the $N-1$ relative coordinates, $\zeta_k = \eta_k - \eta_{k+1}$. Equation (3) then becomes

$$\beta_c \ddot{\zeta}_k + \dot{\zeta}_k + \cos(\varphi_0)\zeta_k = 0 \quad k = 1,2,\ldots,N-1, \tag{4a}$$

$$\beta_c \ddot{\vartheta} + 2\dot{\vartheta} + \cos(\varphi_0)\vartheta = 0. \tag{4b}$$

This transformation decouples all N coordinates in the problem. Further simplification results because all of the relative

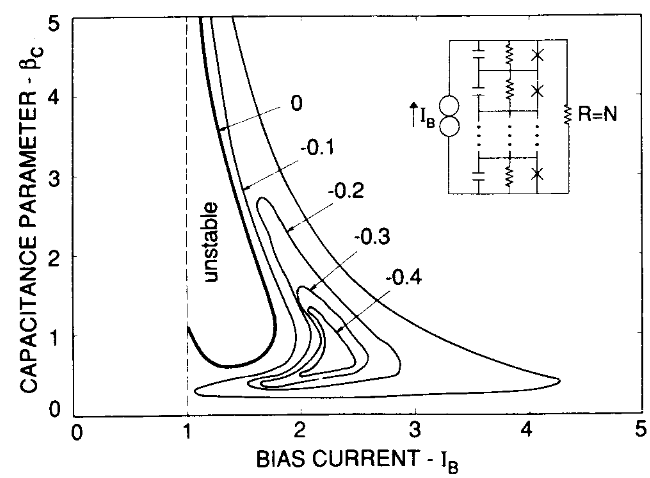

FIG. 1. Contours of the largest real part of the Floquet exponents are plotted as a function of the junction capacitance β_c and the bias current I_B. The coherent solution is unstable for $\mathrm{Re}(\rho) > 0$ and is most stable for the most negative exponents. The plot relates the stability of arbitrarily large arrays of junctions with a load resistance, $R = N$. The circuit we are considering is shown in the inset.

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coordinates obey the same equation. Thus, because of symmetry, the stability analysis of the original N equations reduces to solving the above set of two equations. The coherent solution $\varphi_0(t)$ will remain stable as long as the relative coordinates do not grow. We therefore focus our attention on Eq. (4a).

Equation (4a) arises in many physical problems and has been analyzed thoroughly using Floquet theory.[5] The analysis shows that any solution to this equation can be expressed as a linear combination of two fundamental solutions, $\zeta_a(t)$ and $\zeta_b(t)$, which are specified by the initial conditions: $\zeta_a(0) = 1, \dot{\zeta}_a(0) = 0, \zeta_b(0) = 0, \dot{\zeta}_b(0) = 1$. Since $\cos(\varphi_0)$ is a periodic function, $\zeta_a(t+T)$ and $\zeta_b(t+T)$ must also be solutions to Eq. (4a), which can be expressed in terms of $\zeta_a(t)$ and $\zeta_b(t)$. This leads to the equation

$$\begin{pmatrix} \zeta_a(t+T) \\ \zeta_b(t+T) \end{pmatrix} = \begin{pmatrix} \zeta_a(T) & \dot{\zeta}_a(T) \\ \zeta_b(T) & \dot{\zeta}_b(T) \end{pmatrix} \begin{pmatrix} \zeta_a(t) \\ \zeta_b(t) \end{pmatrix}. \tag{5}$$

The eigensolutions of Eq. (5) are called the Floquet solutions and can be put in the form $\zeta_1 = e^{\rho_1 t}\chi_1(t)$, $\zeta_2 = e^{\rho_2 t}\chi_2(t)$, where $\chi_1(t)$ and $\chi_2(t)$ are periodic functions of period T and $\rho_1 + \rho_2 = -1/\beta_c$. The ρ's are called the Floquet exponents and their real parts determine the stability of the perturbations. They are related to the eigenvalues λ_j of the matrix in Eq. (5) by $\rho_j = \ln(\lambda_j)/T$. If both $\mathrm{Re}(\rho_1) < 0$ and $\mathrm{Re}(\rho_2) < 0$, the perturbations decay and the coherent solution is linearly stable. If either $\mathrm{Re}(\rho_1) > 0$ or $\mathrm{Re}(\rho_2) > 0$, the perturbations grow and the coherent solution is linearly unstable. Finally, if either $\mathrm{Re}(\rho_1) = 0$ or $\mathrm{Re}(\rho_2) = 0$, then the coherent state is neutrally stable and nonlinear terms omitted in writing Eq. (3) determine the stability of $\varphi_0(t)$.

Before considering the general case we examine the two limiting regimes, $\beta_c \to \infty$ and $\beta_c = 0$. In the limit $\beta_c \gg 1$, $\rho_1 + \rho_2 \to 0$, and the condition for linear stability cannot be satisfied. Numerically, we find neutral stability for the circuit being analyzed here. An immediate conclusion from this result is that arrays of junctions with $\beta_c \gg 1$ will not phase lock strongly.

In the highly damped limit, one can take $\beta_c = 0$ and Eq. (4a) can be solved exactly.[6] Direct integration of Eq. (4a) in this case yields

$$\zeta = C_1 \exp\left(-\int_0^t \cos(\varphi_0)\,dt' \right), \tag{6}$$

where φ_0 solves Eq. (2) and C_1 is an arbitrary constant. Differentiating Eq. (2), we can determine that $\cos(\varphi_0) = -2\ddot{\varphi}_0/\dot{\varphi}_0$. Substituting this into Eq. (6) yields the result

$$\zeta = C_1(\dot{\varphi}_0)^2 = C_1\xi^4/[I_B + \cos(\xi t)]^2, \quad \xi = \sqrt{(I_B)^2 - 1}, \tag{7}$$

where we have used the well known solution to Eq. (2) for $\beta_c = 0$.[3] This solution is periodic; it neither grows nor diminishes with time. Therefore, we see that the coherent solution is also neutrally stable for $\beta_c = 0$.

To analyze the stability of the coherent solution in general we have numerically calculated the Floquet exponents associated with the relative coordinate by diagonalizing the matrix in Eq. (5). In Fig. 1 we plot contours of the largest real part of the Floquet exponents as a function of the junction capacitance β_c and the bias current I_B. The heavy line in Fig. 1 is the $\mathrm{Re}(\rho) = 0$ contour, and it separates the stable regions from the unstable regions. The coherent solution is most stable for the most negative exponents. The strongest phase locking occurs for β_c in the range 0.5–1 and I_B in the range 2–2.5. The plot relates the stability of arbitrarily large arrays of junctions where the load resistance is $R = N$. The figure also shows that $\mathrm{Re}(\rho) \to 0$ in the limits $\beta_c \gg 1$ and $\beta_c \ll 1$, in agreement with the analytic results presented above. To our knowledge these results are the first to demonstrate that a series array of Josephson junctions can phase lock coherently with a purely resistive load.

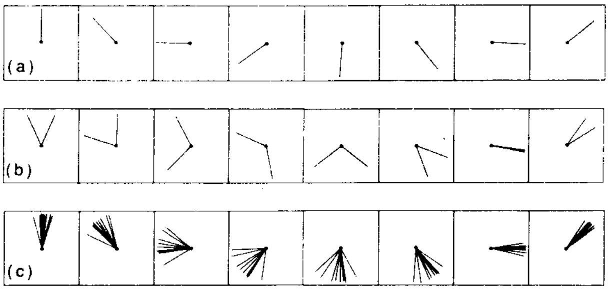

FIG. 2. Series of snapshots of the phases φ_k of a ten-junction array at eight points in the cycle for three sets of array parameters. The phases are measured as angles from vertical. Viewed successively from left to right, the snapshots form a movie of the phase motion. (a) The junctions have phase locked and oscillate coherently, $I_B = 2.3, \beta_c = 0.75$. All ten junctions have the same phase. (b) The period-doubled, symmetry-broken solution that appears just inside the unstable region, $I_B = 1.7$, $\beta_c = 1$. Five junctions have one phase and the other five have another phase. (c) A 10% spread in the critical currents, capacitance parameters, and shunt resistances has been introduced. This simulation shows that the coherent solution can be stable even when the junctions are not identical. $I_B = 2.3$ and the average $\beta_c = 0.75$. Here all ten junctions have different phases.

Figure 2 shows snapshots of the phases φ_k of a ten-junction array at eight points in the cycle for three different sets of parameters. The phases are measured as angles from vertical as they would be in the pendulum analogy to Josephson junctions.[7] Viewed successively from left to right, snapshots form a movie of the phase motion. Figure 2(a) shows the situation where the junctions have phase locked and oscillate coherently. When the stability boundary [the $\mathrm{Re}(\rho) = 0$ contour] is crossed, we observe a period doubling and the resulting solution is not completely coherent. Instead the phases of the junctions divide into two coherent subgroups as shown in Fig. 2(b). In this solution at least $[(N-1)/2]!$ symmetries of the original equations are broken. Further, in the unstable region more bifurcations occur, eventually leading to chaos.

Figure 2(c) demonstrates that the coherent solution can be stable even when the junctions are not identical. In this simulation of a ten-junction array we have introduced a 10% spread in the critical currents, the capacitance parameters, and the shunt resistances. In this figure the lengths of the lines that show the positions of the ten phases are proportional to the critical currents of the junctions.

In conclusion, we have presented a general approach to the stability analysis of series arrays of Josephson junctions that can be readily implemented numerically. Using this analysis, we have demonstrated that arrays with matched resistive loads can phase lock and oscillate coherently. This analysis can be extended to handle arrays with any load and should prove useful in the design of practical arrays.

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[4]It is customary to measure time in units of $\hbar/(2eI_C R_N)$, current in units of I_C, and voltage in units of $I_C R_N$, where I_C is the critical current of the junction and R_N is the shunt resistance. The dimensionless parameter, $\beta_c = (2eI_c R_N^2 C)/\hbar$, characterizes the capacitance of the junction, where C is the capacitance.

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