Dynamical states and stability of linear arrays of Josephson junctions

P. Hadley and M. R. Beasley
Department of Applied Physics, Stanford University, Stanford, California 94305

(Received 20 October 1986; accepted for publication 6 January 1987)

We consider a one-dimensional array of dc-biased Josephson junctions shunted by a load of passive circuit elements. The load serves to couple the ac Josephson effect oscillations in the various junctions, giving rise to dynamical states of the system that do not appear for a single junction. Our results demonstrate two distinct phase-locked states of the array, hysteresis, and chaotic behavior depending on the load and the value of the bias current. Implications of these results for local oscillator applications of such arrays are also discussed.

When the bias current feeding a Josephson junction exceeds the junction's critical current, the supercurrent flowing through the junction oscillates at a frequency that is proportional to the voltage across the junction. This is the familiar ac Josephson effect. Loading the junction with an inductor-capacitor (LC) circuit leads to a competition between the Josephson frequency and the resonant frequency of the LC circuit. The interaction of the two frequencies can result in nontrivial dynamical behavior including bifurcations and chaos.[1,2] In this letter we go beyond single junctions and discuss the behavior of one-dimensional arrays of Josephson junctions. Such arrays have been proposed and analyzed for applications as rf generators and local oscillators in the mm/sub-mm wave range.[3] In the case of an array, an additional effect of the load is to couple the oscillations of the junctions. These interactions of the supercurrent oscillations of the various junctions lead to dynamical states of the array that do not exist for a single junction. As we show, these include two distinct phase-locked states and hysteresis between them, as well as period-doubled and chaotic states. In this letter we use a combination of analytic solutions and numerical simulations to examine this behavior.

To describe the behavior of the individual junctions we employ the well-known resistantly shunted junction (RSJ) model.[4] Using the customary reduced units, the current conservation equations for the N junctions read[5]

$$\dot{\varphi}_k + \sin(\varphi_k) + I_L = I_B. \tag{1}$$

Here φ_k is the quantum phase difference across the k th junction, I_L is the current through the load, and I_B is the bias current. In addition, we have one equation that relates the voltage across the array to the current through the load:

$$V = \sum_{k=1}^{N} \dot{\varphi}_k = \mathscr{L}\{I_L\}, \tag{2}$$

where $\mathscr{L}$ is a linear integrodifferential operator that depends on the impedance of the load.

We have performed numerical integration of these equations using a fourth-order Runge–Kutta routine.[6] The simulations show that there are two phase-locked solutions, an in-phase solution and an antiphase solution, which we describe below. Depending on the load chosen, neither, one, or both of these phase-locked solutions can be stable for a given bias current. Much of the behavior of the arrays can be understood in terms of these two solutions.

In the in-phase case each junction oscillates with the same frequency and phase, that is, $\varphi_k = \varphi_j$. When all of the phase differences are the same, the system of equations, (1) and (2), reduce to the equation for a single junction shunted by a load whose current-voltage characteristic is given by $V = \mathscr{L}(I_L)/N$. Thus, setting aside questions of stability, the in-phase case reduces to the simpler and previously studied problem of a single junction with a suitably scaled load. This solution has stimulated interest for rf generator applications because such a phase-locked array of N junctions provides N times the output voltage and N^2 times the output power as does a single junction.

The other phase-locked solution that we observe for arrays of two or three junctions is an antiphase solution in which all of the junctions undergo the same T periodic oscillation but with staggered phases:

$$\varphi_k(t) = \varphi_0(t + Tk/N) \quad k = 1,2,...N-1. \tag{3}$$

The solution $\varphi_0(t)$ is a periodic function of time so we can write it as

$$\varphi_0(t) = \sum_{p=1}^{\infty} A_p e^{i2\pi pt/T}. \tag{4}$$

Thus in the antiphase solution the voltage across the array is

$$V = \sum_{k=1}^{N} \dot{\varphi}_0\left(t + \frac{Tk}{N}\right) = \sum_{p=N,2N,3N,...}^{\infty} \frac{i2\pi Np}{T} A_p e^{i2\pi pt/T}. \tag{5}$$

Hence in the antiphase case the load current has frequency components only at $(2\pi N/T, 4\pi N/T,...)$; all other frequency components vanish. The amplitudes of the nonzero harmonics decrease with increasing bias current, so for high bias currents little ac current flows through the load. With a small current flowing through the load, there is only weak coupling of the junctions. In the limit of vanishingly small load current, the above equations for the array of junctions decouple into N independent equations, each identical to the equation for a single junction with no external load. The actual behavior of an array approaches this limiting solution as the bias current is increased and the ac Josephson oscillations become increasingly sinusoidal. When there are more than three junctions in the array, the antiphase solution still exists, by which we mean that we observe a solution where the sum of the fundamental Josephson oscillations adds to zero. However, for more than three junctions the phase distribution is more complicated than the simple staggering found for two and three junctions.

Our simulations show that the in-phase solution is stable when the impedance of the load is inductive at the funda-

 0003-6951/87/100621-03$01.00

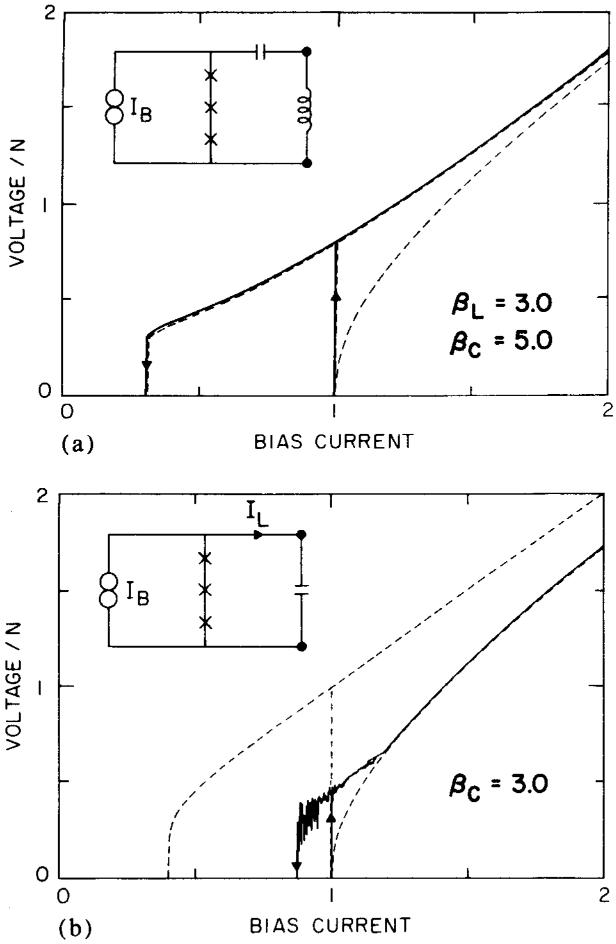

FIG. 1. (a) Current-voltage curve for a three-junction array with an inductive load, $\beta_L = 3$, and a blocking capacitor, $\beta_C = 5$. The dashed lines represent the in-phase and the limiting antiphase solutions. The curve shows that the in-phase solution is stable for an inductive load. (b) Current-voltage curve for a three-junction array with a capacitive load, $\beta_C = 3$. This figure shows that the antiphase solution is stable at high bias for a capacitive load.

mental Josephson frequency and that the antiphase solution is stable when it is capacitive. This confirms the perturbation calculations for high bias currents of Jain *et al.*[3] There are also regimes of more complex dynamics, where neither phase-locked solution is stable, which exist for arrays biased close to their critical currents. These results are discussed below.

In Fig. 1 we compare the observed behavior for inductive and capacitive loads. The solid line in Fig. 1(a) is the current-voltage curve for a three-junction array shunted by an inductive load with a blocking capacitor that is included to direct the dc current through the junctions. The dashed lines in the figure are single-junction solutions that represent the in-phase and the limiting antiphase solutions described above. The in-phase solution was obtained numerically, and the limiting antiphase solution is the well-known current-voltage curve of a single junction in the RSJ mode. With an inductive load, only the in-phase solution is stable and the array responds like a single junction with an external load of $\mathscr{L}(I_L)/N$ over the entire range of bias currents.

Figure 1(b) shows the same curves for a three-junction array shunted by a capacitor. In this case our simulations show that the antiphase solution is stable for high bias currents and is very close to the limiting independent junction solution. However, for bias currents near the critical current the antiphase solution goes unstable and there is a kink visible in the current-voltage curve. Below the kink neither phase-locked solution is stable. By taking phase portraits of the motion on both sides of the kink, we have determined that it signals a symmetry breaking bifurcation. The symmetry that is broken is a permutation symmetry; the system is unchanged under the interchange of any two junctions. Figure 2 shows some phase portraits of the motion. In these portraits a closed loop corresponds to periodic motion. The sequence of phase portraits shows the symmetry breaking followed by a cascade of period-doubling bifurcations and

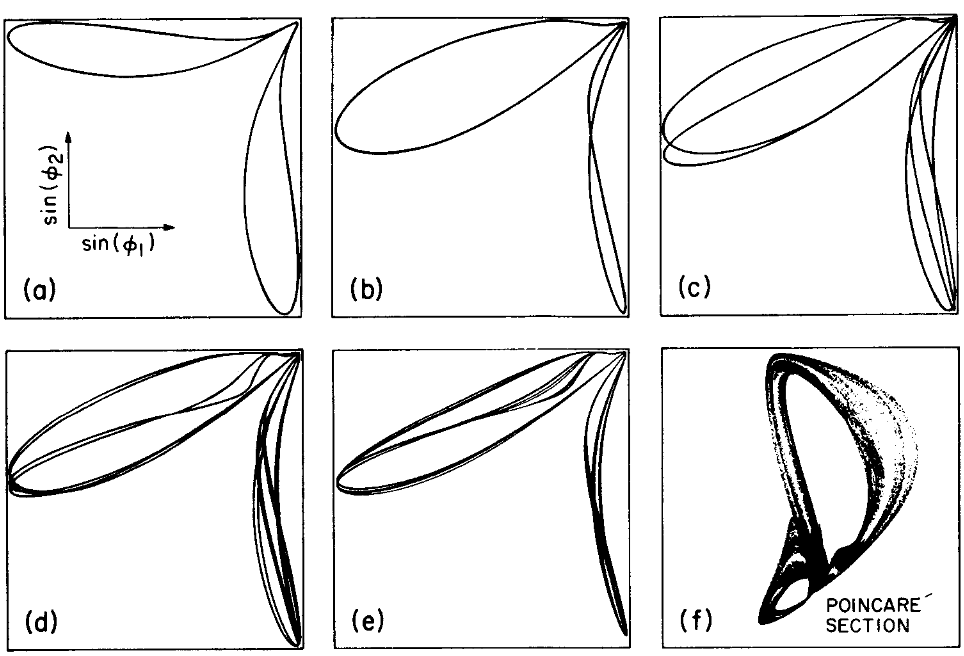

FIG. 2. (a)–(e) are phase portraits of the motion, in the vicinity of the bifurcation seen in 1(b), showing symmetry breaking followed by a period-doubling cascade. The portraits are projections of the trajectory in phase space onto the $\sin(\varphi_1)$ vs $\sin(\varphi_2)$ plane. (f) is a Poincaré section of chaotic motion showing the fractal structure of the strange attractor.

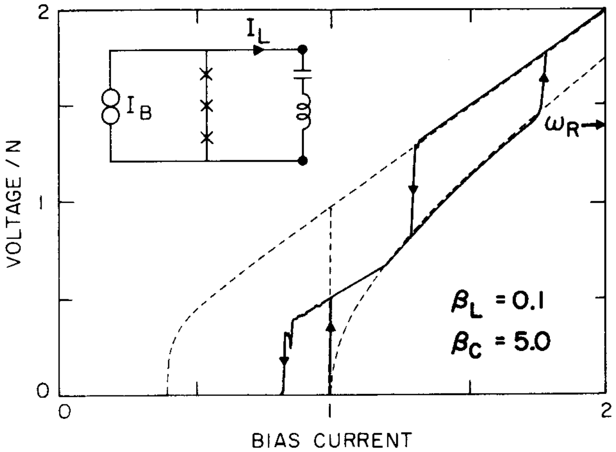

FIG. 3. Current-voltage curve for a three-junction array shunted by a series LC circuit: $\beta_C = 5$; $\beta_L = 0.1$ The dashed lines represent the in-phase and the antiphase solutions. The arrow on the right indicates the value of $\omega_R = (\beta_C\beta_L)^{-1/2} = \sqrt{2}$. This figure shows the hysteresis between the in-phase and antiphase solutions.

eventually leading to chaos. Figure 2(f) is a Poincaré section of the chaotic motion that shows the fractal structure of the strange attractor. Chaotic behavior is observed for arrays with capacitive loads, $N\beta_c > 1$, biased in the vicinity of the critical current. The range of bias currents over which chaos is observed increases with increasing β_c.

When the array is shunted by a series LC circuit, as in Fig. 3, the reactance of the load at the Josephson frequency goes from capacitive to inductive as the bias current is increased. In this case the simulations show that the in-phase and antiphase solutions exchange stability in a way that produces a hysteresis loop in the current-voltage curve. Beginning at low bias current the antiphase solution remains stable, as the bias current is increased, until the fundamental Josephson frequency of the antiphase solution exceeds the LC resonant frequency $[\omega_R = (\beta_C\beta_L)^{-1/2}]$ and the load becomes inductive. At that point the system jumps to the in-phase solution and the fundamental Josephson oscillation jumps to a higher frequency. This makes the load look more inductive. The in-phase solution will thus remain stable until the bias current is decreased to the point at which the fundamental Josephson frequency of the in phase is less than the LC resonant frequency, and the load looks capacitive once again. Preliminary results show that for a parallel LC load, the in-phase and antiphase solutions still exchange stability at the LC resonant frequency but that there is no hysteresis. The difference between the series and parallel cases is that a series LC circuit becomes a short at resonance whereas a parallel LC circuit becomes an open circuit. Thus there is no coupling between the junctions at resonance for the parallel case, and the transition to the antiphase solution is continuous.

From our simulations we conclude that the minimum number of junctions needed in an array to observe the two phase-locked solutions and hysteresis is two. The in-phase and antiphase solutions exist for arrays with any number of junctions. We have observed these solutions in simulations with up to 50 junctions, but we have not made systematic studies of their stability for arrays with more than three junctions.

Finally, our results demonstrate that some care must be exercised in constructing Josephson junction local oscillator arrays when capacitive circuit elements are present, for example, due to the junction capacitance itself. As shown by Jain *et al.* and confirmed over a wide range of conditions by our simulations, in-phase locking requires that the load look inductive at the operating frequency. However, our results show that when tuning the load to look inductive, consideration must also be given to the effects of hysteresis or the reduction of the coupling strength between the individual junctions near any resonant frequency of the load.

We are pleased to acknowledge many interesting and useful discussions with John Bourg. This work was supported by the National Science Foundation.

[1]R. F. Miracky, J. Clarke, and R. H. Koch, Phys. Rev. Lett. **50**, 856 (1983).
[2]A. D. Smith, R. A. Sandell, A. H. Silver, and R. W. Simon, Bull. Am. Phys. Soc. **31**, 419 (1986).
[3]For a comprehensive review see, A. K. Jain, K. K. Likharev, J. E. Lukens, and J. E. Sauvageau, Phys. Rep. **109**, 310 (1984).
[4]See, for example, T. Van Duzer and C. W. Turner, *Principles of Superconductive Devices and Circuits* (Elsevier, New York, 1981), p. 170.
[5]It is customary to measure time in units of $\hbar/(2eI_CR_N)$, current in units of I_C, and voltage in units of I_CR_N, where I_C is the critical current of the junction and R_N is the shunt resistance. The dimensionless parameters, $\beta_C = (2eI_CR_N^2C)/\hbar$ and $\beta_L = (2eI_CL)/\hbar$, characterize the capacitance and the inductance of the load.
[6]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (U.S. Government Printing Office, Washington, DC, 1972), p. 896.